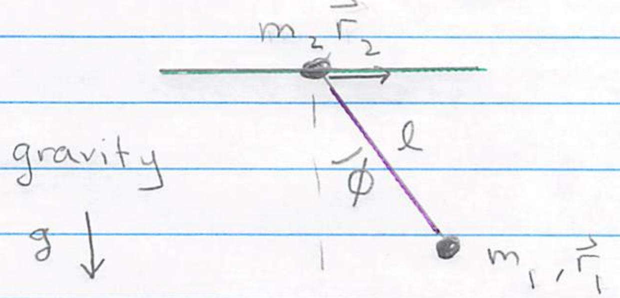
An example:

- Consider a pendulum on a wire. The pendulum is free to move on the wire. determine the EOM. The rod has length ℓ and is light, the two masses are m_1 and m_2



- It pays to use the right coordinates. There are no forces in the x-direction, so the x-position of the CM will move with a constant velocity. We should take x_{cm} as one of our coordinates:

$$\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

center of mass relative coords

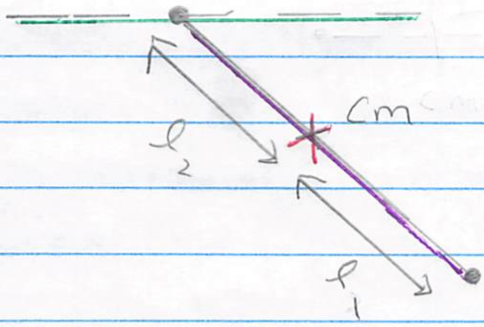
- Then, in homework show

$$T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^{\,2}$$

with $M \equiv m_1 + m_2$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2} \quad \leftarrow \text{"reduced mass"}$$

For later use we note:

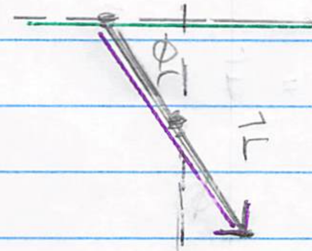


$$l_1 \equiv \frac{m_2 l}{m_1 + m_2}$$

$$l_2 = \frac{m_1 l}{m_1 + m_2}$$

Now:

$$\vec{r} = (r\sin\phi, -r\cos\phi)$$



And then

$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

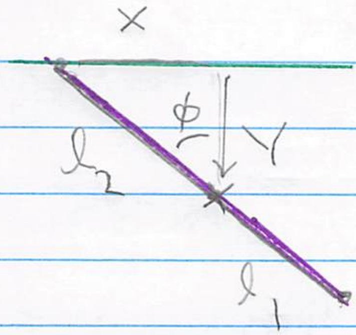
- So the kinetic energy for two particle is quite generally

$$\boxed{T = \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)}$$

where $\vec{R} = (X, Y)$

- For the problem at hand $r = l$, and it is fixed length $\vec{r}$

- When ϕ changes then the height of cm Y also changes. From geometry we have...



$$Y = -l_2 \cos\phi$$

$$\dot{Y} = l_2 \sin\phi \, \dot{\phi}$$

So

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \, l_2^2 \sin^2\phi \, \dot{\phi}^2 + \frac{1}{2} \mu \, l^2 \dot{\phi}^2$$

$U = -mgY$ just gravity folks

$$= -m g \, l_2 \cos\phi$$

- So finally $L = T - U$

$$\boxed{L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} I_{eff}(\phi) \, \dot{\phi}^2 + m g \, l_2 \cos\phi}$$

where $I_{eff}(\phi) \equiv m \, l_2^2 \sin^2\phi + \mu \, l^2$ is a kind of effective moment of inertia.

- The EOM are from Euler Lagrange

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \cancel{\frac{\partial L}{\partial x}} \to 0$$

We note that the Lagrangian doesn't depend on X. We say it is a "cyclic" coordinate

The corresponding momentum $p_x = \frac{\partial L}{\partial \dot{x}}$ is constant.

i.e.

$$\frac{\partial}{\partial t}(M\dot{x}) = 0 \qquad M\dot{x} = \text{const}$$

as we anticipated

- The EOM for ϕ is:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{\partial}{\partial t}\left(I_{eff}(\phi)\,\dot{\phi}\right) = \frac{1}{2}\frac{\partial I_{eff}}{\partial \phi}\dot{\phi}^2 - mg\,l_2 \sin\phi$$

change in angular momentum per time

Torque by the plane holding up the masses

torque by gravity

Getting the EOM with Newton Law's would be quite difficult. It is doable with Lagrangians!