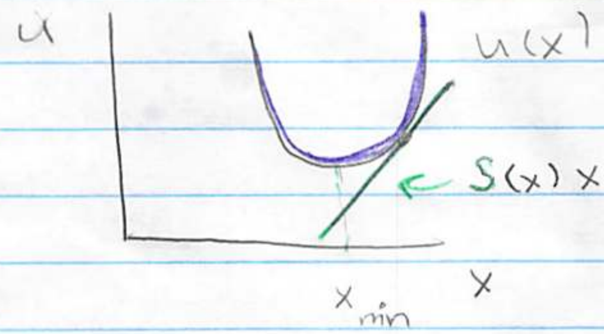
More about Legendre Transforms (LT)

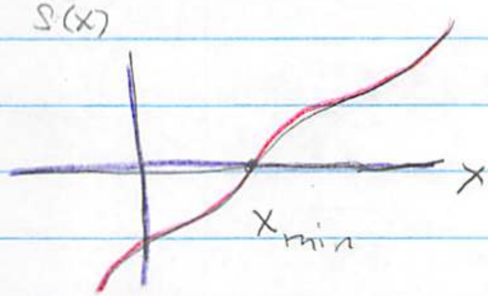
- Take a function, covex, $U(x)$:



- The legendre transform Characterizes $U(x)$ by its slopes $S(x)$. It is used throughout physics

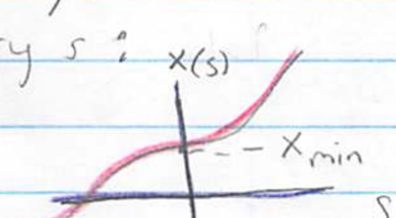
- Then for every value of x - there is a slope $S(x)$:

$$S(x) = \frac{\partial U(x)}{\partial x}$$



For convex functions the slope changes monotonically with x, and thus there is one x for every s:

$$x(s)$$



- The Legendre transform is a function of the slope, $V(s)$:

$$V(s) = s\,x(s) - U(\,x(s)\,)$$

- Alternatively we define the Legendre Transform as

definition

$$\boxed{V(s) \equiv \underset{x}{\text{extrm}}\{sx - U(x)\} \leftarrow \text{extremize } sx - U \text{ by changing } x}$$

- To show that this works, we find the place where the x-derivative of $sx - U(x)$ is zero:

$$s - \frac{\partial U(x)}{\partial x} = 0 \Rightarrow s = \frac{\partial U}{\partial x}(x) \Rightarrow \underset{\text{solve for } x}{x = x(s)}$$

So $V(s) = s\,x(s) - U(x(s))$ as before.

- The LT is a map from coordinates to slope space

$$\boxed{x, U(x) \Longleftrightarrow s, V(s) \leftarrow \text{characterizes } U \text{ by its slope}}$$

We can undo the Legendre transform by going in reverse

$$U(x) = \min_{s}(sx - V(s))$$ (Prove me!)

Note also

$$dU = s\,dx \Rightarrow \boxed{\begin{aligned} dU/dx &= s(x) \\ dV/ds &= x(s) \end{aligned}}$$

$$d(sx - U) = x\,ds \Rightarrow$$

• In mechanics:

$$\underline{v,\ L(x,v) \iff p,\ H(x,p)}$$

where

$$\boxed{H(x,p) = \underset{v}{\mathrm{extrm}} \{ pv - L(x,v) \}}$$

And

$$dL = p\, dV + \ldots \text{ spectors } \partial L/\partial x,\ \partial L/\partial t \text{ etc}$$

$$dH = v\, dp - \text{spectators}$$

• Homework:

a) Consider a potential $U(x)$, where x is the coordinate of a particle. Suppose that an external force is appleed (of magnitude f_0) in the positive x-dipection. Show that Legendre Transform of $U(x)$ is related to the minimum value of the (new) potential energy function for a given force. The minium value is a function of the applied force.

b) What is the legendre transform of $\frac{1}{2} k (x - x_0)^2$ and interpret using (a).