Noether Theorem

- Codifies the relation between a continuous symmetry and conservation laws.

- For definiteness take a Lagrangian of two particles under a central force

$$L = \frac{1}{2} m \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

- The Lagrangian and action is invariant under shifts of the coordinates. The symmetry is homogeneity of space

$$\vec{r}_a \longrightarrow \vec{r}_a + \varepsilon \vec{n} \qquad a = 1, 2$$

For all particles a constant infinitessimal shift. For definiteness take $\vec{n} = \hat{x}$

Lagrangian is Invariant

$$L \longrightarrow L$$

Let us write generally

$$\vec{r}_a \longrightarrow \vec{r}_a + \delta_S \vec{r}_a \qquad \text{e.g.} \quad \delta_S \vec{r}_1 = (\varepsilon, 0, 0)$$

a symmetry variation

- The symmetry variations are not as general as the EOM variations we considered earlier, and do <u>not</u> vanish at the ends

Then define:

$$\boxed{\delta S[\vec{r}, \delta\vec{r}] \equiv S[\vec{r}_a + \delta\vec{r}_a] - S[\vec{r}_a]}$$

The transformation is a symmetry if

☆ $\boxed{\delta S[\vec{r}, \delta_s\vec{r}] = 0}$ (L-is unchanged)

- Note that $\vec{r}$ does not need to satisfy the EOM. If $\vec{r}_a$ does satisfy the EOM we put a bar under it $\underline{\vec{r}}_a$, and say that it is "onshell"
- Now consider a general variation:

$$S[r+\delta r] = S[r] + \int dt \, \frac{\partial L}{\partial \vec{r}_a} \delta\vec{r}_a + \underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_a}}_{\text{this is } \vec{p}_a} \underbrace{\frac{d(\delta\vec{r}_a)}{dt}}_{\text{integrate by parts}}$$

- So a general variation gives

$$\delta S[r, \delta r] = \vec{p}_a \cdot \delta\vec{r}_a \Big|_{t_1}^{t_2} + \int dt \underbrace{\left(\frac{\partial L}{\partial \vec{r}_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_a}\right)}_{\text{this vanishes if } \vec{r} \text{ satisfies the EOM}} \cdot \delta\vec{r}_a$$

So for $\underline{\vec{r}}$ onshell

☆ $$\boxed{\delta S[\underline{r}, \delta r] = \vec{p}_a \cdot \delta\vec{r}_a \Big|_{t_1}^{t_2}}$$

- So we may restrict ourselves to symmetry variation. Comparing $\bigstar$ to $\bigstar$

$$\delta S[\underline{\vec{r}}, \delta_s \vec{r}] = 0 = \vec{p}_a \cdot \delta_s \vec{r}_a \Big|_{t=t_2} - \vec{p}_a \cdot \delta_s \vec{r}_a \Big|_{t=t_1}$$

- Yielding a conservation Law, of Q

$$\boxed{Q = \vec{p}_a \cdot \delta_s \vec{r}_a}$$

- Since in our first example

$$\delta_s \vec{r} = \varepsilon \vec{n}$$

momentum in $\vec{n}$ direction

Then

$$Q = p_a \cdot \delta_s \vec{r}_a = \sum_a \vec{p}_a \cdot \varepsilon \vec{n} = \varepsilon \vec{n} \cdot \sum_a \vec{p}_a$$

So the conservation law evaluates

$$\underline{\varepsilon \vec{n} \cdot \left(\sum_a \vec{p}_a \Big|_{t_2} - \sum_a \vec{p}_a \Big|_{t_1} \right) = 0}$$

Or since $\vec{n}$ was arbitrary

$$\sum_a \vec{p}_a \Big|_{t_2} - \sum_a \vec{p}_a \Big|_{t_1} = 0$$