Oscillations and Normal Modes

- Let us consider the Lagrangian:

$$L = \frac{1}{2} m_{ij}(q)\, \dot{q}^i \dot{q}^j - U(q)$$

- Expand the potential near a minimum

$$q^i = q_0^i + x^i$$

← small, not necessarily cartesian.

So

$$U(q) = U(q_0) + \frac{\partial U}{\partial q^i} x^i + \frac{1}{2} \underbrace{\frac{\partial U}{\partial q^i \partial q^j}}_{\equiv\, k_{ij}} x^i x^j + \ldots$$

This is zero at a minimum.

↖ matrix of spring consts.

- Then

$$L = \frac{1}{2} m_{ij} \dot{x}^i \dot{x}^j - \frac{1}{2} k_{ij} x^i x^j$$

where the matrices m_{ij} and k_{ij} are evaluated at q_0

- The equation of motion is

$$m_{ij} \ddot{x}^j = -k_{ij} x^j$$

- The way we can solve this is to try a solution

(this describes an oscillation with one frequency, i.e. a normal mode)

$$x^j(t) = E^j e^{-i\omega t}$$

constant vector (it will be real)

So this becomes

$$k_{ij} E^j = \omega^2 m_{ij} E^j$$

- So with a matrix notation

$$K = \begin{pmatrix} k_{ij} \end{pmatrix} \qquad M = \begin{pmatrix} m_{ij} \end{pmatrix} \qquad \vec{E} \equiv \begin{pmatrix} E^j \end{pmatrix}$$

we have

positive definite real and symmetric matrix M

$$\boxed{K \vec{E} = \omega^2 M \vec{E}}$$

This is a (generalized) eigenvalue problem for the eigenvectors $\vec{E}$ and the eigenvalues ω^2.

- We will describe this e-value problem in the next section. Here we procede with a simple method that always works:

$$\star \quad (\mathbb{K} - \omega^2 \mathbb{M})\,\vec{E} = 0$$

- This has non-trivial solutions ($\vec{E} \neq 0$) only when

$$\det(\mathbb{K} - \omega^2 \mathbb{M}) = 0$$

This characteristic polynomial has roots

$$\lambda_a = \omega_a^2 \quad \text{with} \quad a = 1 \ldots N$$

Once you know λ_a you can substitute into $\star$, and determine the eigenvector for each a, $\vec{E}_a$ with $a = 1 \ldots N$

- The general solution is an expansion

$$x^j(t) = \sum_a C_+^a E_a^j e^{-i\omega_a t} + C_-^a E_a^j e^{+i\omega_a t}$$

general solution

Now you should adjust the coefficients C_+^a and C_-^a to reproduce the initial conditions for x:

$x^j(0)$ and $\dot{x}^j(0)$

The Generalized E-Value Problem

- More common and basically the same as the ordinary E-value problem

$$A \cdot \vec{E} = \lambda W \vec{E}$$

matrix

real positive definite weight matrix, M in our case.

- To find λ just do the usual thing:

$$\det(A - \lambda W) = 0 \leftarrow \text{solve}$$

- Let A be a hermitian matrix

① The eigenvalues are real

② The eigenvectors are orthogonal with W as a weight

$$(\vec{X}, \vec{Y}) = X^{*i} Y^j \leftarrow \text{Inner product}$$

$$(\vec{X}, W\vec{Y}) = X^{*i} W_{ij} X^j \leftarrow \text{Weighted inner product.}$$

So, we mean:

$$(E_a, WE_b) \equiv E_a^{*i} W_{ij} E_b^j \equiv E_a^\dagger W E_b = \delta_{ab}$$

Inner product of two E-vectors with a weight matrix W.

<u>Proof</u>

• Reality

$$\star \quad (\vec{E}, A\vec{E}) = \lambda(\vec{E}, W\vec{E})$$

real, W is real and symmetric

But, $(E, AE) = (AE, E) = (E, AE)^* = \text{real}$

A self-adjoint (hermitian) matrix

So $\star$ can be satisfied for λ real

• Orthogonality: Use $AE = \lambda W$

So $\quad (E_2, AE_1) = \lambda_1 (E_2, WE_1)$

And $\quad (E_1, AE_2)^* = \lambda_2 (E_1, WE_2)^*$

Then using $(E_1, AE_2)^* = (E_2, AE_1)$ and $(E_1, WE_2)^* = (E_2, WE_1)$, and subtracting yield's

$$0 = (\lambda_1 - \lambda_2)\,\langle E_2, WE_1 \rangle$$

So for $\lambda_1 \neq \lambda_2$

$$0 = (E_2, WE_1)$$

Return to Mechanics

- Let us express our Lagrangian in terms of the Eigenvector coordinate System:

$$x^j(t) = \sum_a E^j_a \, X^a(t) \qquad K \cdot \vec{E}_a = \omega^2_a \, m \vec{E}_a$$

e-values

mass matrix is weight W matrix

Or

$$\big(x(t)\big) = \big(E_1\big) X^1(t) + \big(E_2\big) X^2(t) + \cdots$$

- We can determine $X^a(t)$ using the orthogonality of the E_a

$$X^a = (E_a, m\, x(t)) = E_a^{\,i} \, m_{ij} \, x^j(t)$$

We used

$$(E_a, m E_b) = \delta^a_{\ b}$$

- The potential energy is

$$U = \frac{1}{2} k_{ij} x^i x^j = \frac{1}{2} \sum_{ab} X^a X^b \, E^i_a k_{ij} E^j_b$$

e-value ↕ eqution

$$= \frac{1}{2} \sum_{ab} X^a X^b \, \underbrace{E^i_a \omega^2_b m_{ij} E^j_b}_{\omega^2_b \delta_{ab}}$$

$$= \frac{1}{2} \sum_a \omega^2_a X^a X^a$$

- The Kinetic energy is similar:

$$\frac{1}{2} m_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} \sum_{ab} \dot{X}^a \dot{X}^b \, \underbrace{E^i_a m_{ij} E^j_b}_{\delta_{ab}}$$

$$= \frac{1}{2} \sum_a (\dot{X}^a)^2$$

- So the Langrian is

$$L = \sum_a \frac{1}{2} \dot{X}_a^2 - \omega_a^2 X_a^2$$

this is independent oscillators with natural frequencies ω_a