Parametric Resonance

- Consider an oscillator with a changing resonant frequency

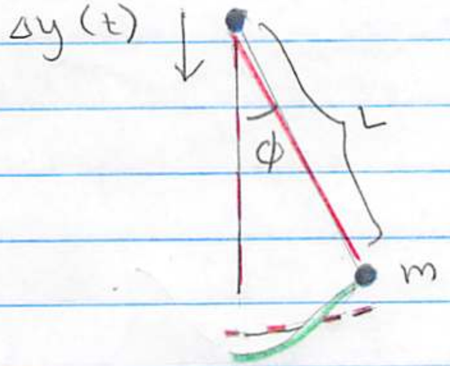
$$\frac{d^2 q}{dt^2} + \omega_0^2 (1 + h \cos \Omega t)\, q = 0$$

$\Omega = 2\omega_0 + \varepsilon$

small

- This model arises in an enormous number of applications
  - Stability of the paul trap for charged ions
  - reheating after inflation
  - Stability of helicopter blades
  - hawking radiation . . .

- As a model consider a pendulum with a oscillating support $\Delta y = -y_0 \cos \Omega t$ with $\Omega \simeq 2\omega_0$, so that the support "drops" the mass at the top of the arc ϕ_{max}, and pulls the mass back up at $\phi \simeq 0$



$$\ddot{\phi} + \omega_0^2 \left(1 + \frac{y_0 \Omega^2}{L} \cos \Omega t\right) \phi = 0$$

- First consider the unperturbed oscillator and calculate the tension as a function of time in the rod

- The tension is smaller than mg at ϕ_{max}

$$T_{min} = mg\cos\phi_{max} = mg\,(1 - \phi^2_{max}/2)$$

$$= mg - \frac{E}{L}$$ ← energy in oscillator $\propto \phi^2_{max}$

- But larger than mg at the bottom:

$$T_{max} = mg + \frac{mv^2}{L} = mg + 2E/L$$

"centrifugal force"

$E = \frac{1}{2}mv^2$

- So if the mass is dropped by Δy at ϕ_{max}, and pulled up by Δy at $\phi \simeq 0$ the external work done in half a period $T_0/2$ is

$$W = (T_{max} - T_{min})\,\Delta y = 3\frac{E}{L}\Delta y$$

Since this is positive, the energy will constantly grow:

$$\dot{E} = \frac{\Delta E}{T_0/2} = 6\,\frac{\Delta y}{L}\,\frac{E}{T_0}$$

leading to exponential growth in the oscillator energy by the external work.

- Now consider what happens if the modulation frequence differs from the resonance condition

$$\underbrace{\Omega}_{\text{modulation frequency}} = 2\omega_0 + \underbrace{\varepsilon}_{\text{detuning parameter}}$$

if detuned the external work will not always be positive

- If the modulation amplitude is large enough it can overcome the detuning

$$\omega_0^2(1 + \underbrace{h}_{\text{modulation amplitude}} \cos\Omega t)$$

Causing exponential growth. We will see: (1) that the condition is

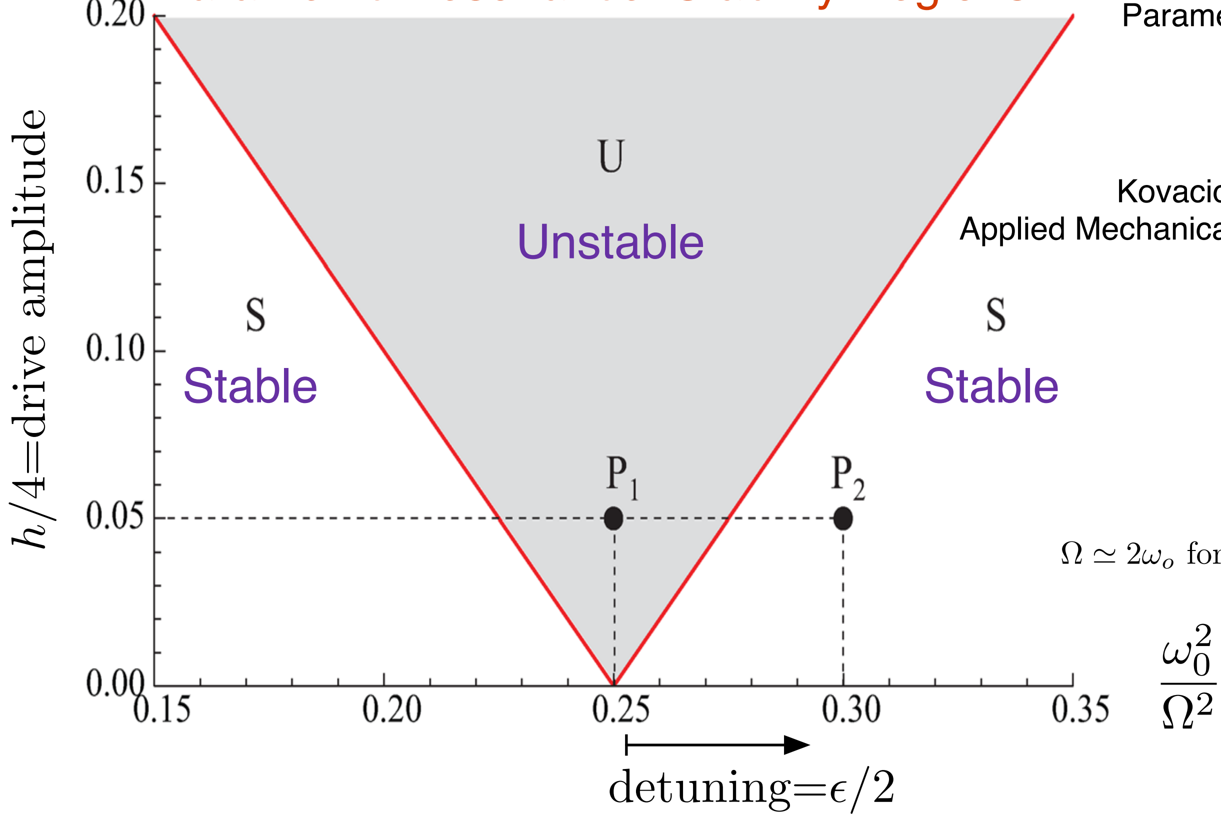
$$\boxed{|\varepsilon| < \frac{1}{2}h\omega_0}$$

The stability regions are shown on the next slide (click me)

(2) The growth rate in amplitude is

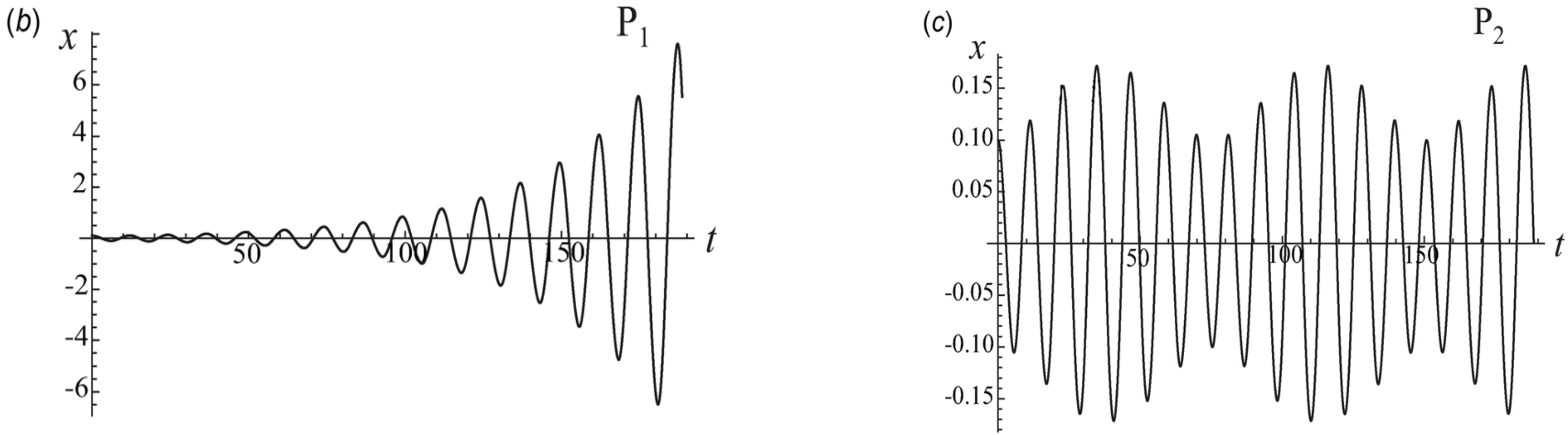
$$\boxed{\lambda = \sqrt{\left(\frac{h\omega_0}{2}\right)^2 - \varepsilon^2}}$$

Parametric Resonance: Stability Regions



Kovacic, Rand, Sah
Applied Mechanical Review, 70, 020802-1

$\Omega \simeq 2\omega_o$ for exponential growth



Unstable oscillations at P1

Stable oscillations at P2

Mathematical Analysis of Parametric Resonance

- We study:

☆ $\ddot{q} + \omega_0^2 (1 + h\cos\Omega t)\, q = 0$

with $\Omega = 2\omega_0 + \varepsilon \equiv 2\omega$. As always we work with a rotating wave / slow roll / secular perturbation theory / WKB approximation (all these are basically the same!)

$$q(t) = q^{(0)}(t) + q^{(1)}(t)$$

with

$$q^{(0)} = \mathrm{Re}\left[A(t)\, e^{-i\omega t} \right]$$

$$= \underbrace{a(t)}\cos\omega t + \underbrace{b(t)}\sin\omega t$$

slow functions of time

- Then note:

(1) $\ddot{q}^{(0)} = -\omega^2 q^{(0)} + \left[-2\omega \sin\omega t\, \dot{a} + 2\omega\cos\omega t\, \dot{b} \right] + (\text{small})^2$

(2) $(\cos 2\omega t)(\cos\omega t) = \frac{1}{2}\left(e^{i2\omega t} + e^{-2i\omega t}\right) \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$$= \frac{1}{2}\cos 3\omega t + \frac{1}{2}\cos\omega t$$

(3) $\cos 2\omega t \sin \omega t = \frac{1}{2} \sin 3\omega t - \frac{1}{2} \sin \omega t$

(4) $(\omega_0^2 - \omega^2)\, q^{(0)} \simeq -\omega \varepsilon\, q^{(0)}$

- Substituting into ☆

this can be set to ω_0 at first order

$$\ddot{q}^{(1)} + \omega_0^2 q^{(1)} + \underline{(2\dot{b} - a\varepsilon + \tfrac{1}{2} h\omega\, a)}\, \omega \cos \omega t$$

$$\underline{(-2\dot{a} - b\varepsilon - \tfrac{1}{2} h\omega\, b)}\, \omega \sin \omega t$$

$$+ \; 3\omega t \text{ terms} = 0$$

- As is usual the $q^{(1)}$ will solve the 3ω terms. The underlined (secular) terms must be set to zero to avoid secular divergences

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon + h\omega_0/2 \\ -\varepsilon + \frac{h\omega_0}{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- This system of equations is solved by finding the eigenvectors and evalues

$$\boxed{\lambda_\pm = \pm \sqrt{-\varepsilon^2 + \left(\frac{h\omega_0}{2}\right)^2}}$$ and corresponding vectors

E_+ and E_-

The Solution is

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_+ e^{\lambda_+ t} \begin{pmatrix} E_+ \end{pmatrix} + c_- e^{\lambda_- t} \begin{pmatrix} E_- \end{pmatrix}$$

(1) If the drive amplitude is large enough:

$$\boxed{|\varepsilon| < \frac{h\omega}{2}},$$

- It overwhelms the detuning, and the eigen values are real with $\lambda_+ > 0$. The oscillation explodes exponentially. This is the point P_1 on the next slide (click me)

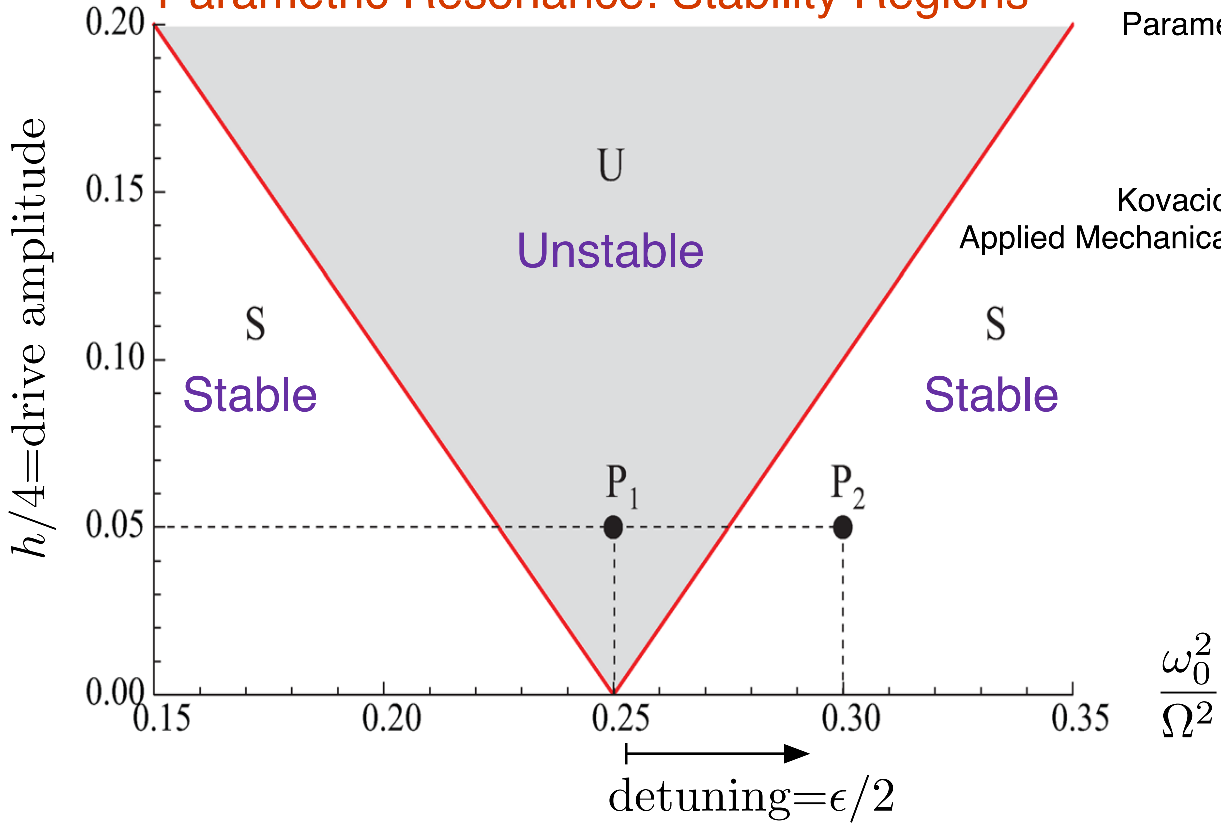
(2) If the drive frequency Ω is not too close to $2\omega_0$. Then the e-values are imaginary:

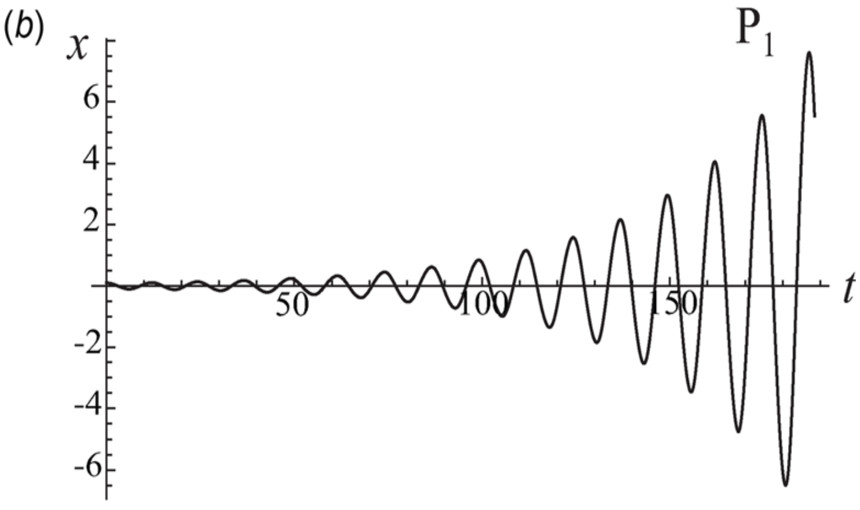
$$\lambda_\pm = \pm i \sqrt{\varepsilon^2 - \left(\frac{h\omega}{2}\right)^2} \qquad \text{for} \qquad \frac{h\omega_0}{2} < |\varepsilon|$$

- The oscillations are stable. The slowly oscillating $a(t)$ and $b(t)$ will envelope the more rapid $\cos\Omega t$ oscillations. This is the point P_2 on the next slide. Click me

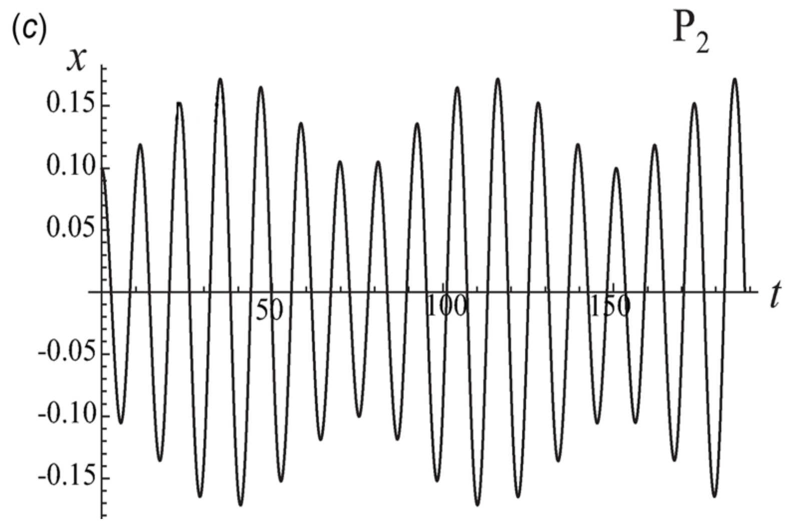
Parametric Resonance: Stability Regions

Kovacic, Rand, Sah
Applied Mechanical Review, 70, 020802-1





Unstable oscillations at P1



Stable oscillations at P2