Retarded Green Function

- Our goal is to find a specific solution for a general force $F(t)$

$$\star \quad \underbrace{\left[m\frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right]}_{\text{I call this operator } \mathcal{L}_t \text{ below}} x_s(t) = F(t)$$

Let us solve instead

$$\left[m\frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right] G_R(t,t_0) = \delta(t-t_0)$$

- $G_R(t,t_0)$ is the displacement at time t, due to an impulsive force at t_0. $\mathcal{L}_t G(t,t_0) = \delta(t-t_0)$.

- The specific solution is

$$(1) \qquad X_s(t) = \int_{-\infty}^{\infty} G_R(t,t_0)\, F(t_0)\, dt_0$$

- Then this clearly satisfies the Equation ($\star$)

$$\mathcal{L}_t\, x_s(t) = \int_{-\infty}^{\infty} \mathcal{L}_t\, G(t,t_0)\, F(t_0)\, dt_0$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\, F(t_0)\, dt_0 = F(t)$$

- We are interested in the retarded Green function. This means that the displacement at a time t, before the force acts, is zero:

$$G_R(t, t_0) = 0 \quad \text{for} \quad t < t_0$$

- We will describe two methods for finding G_R

① A Fourier Space Method (Not Shown)

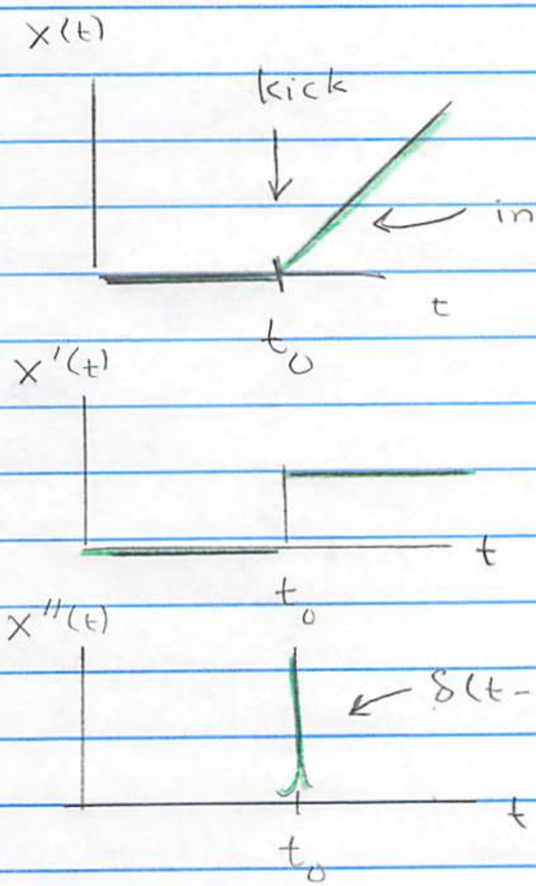
(I give some notes online about if interested)

② A coordinate Space Method

Coordinate Space for $G_R(t,t_0)$

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right] G_R(t,t_0) = \delta(t-t_0)$$

- Recalling that $G_R(t,t_0)$ is the displacement due to a kick at t_0, we are looking for a solution that looks like this near t_0



- To find the initial velocity after the kick we integrate from $t = t_0 - \varepsilon$ to $t = t_0 + \varepsilon$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} dt \left[m\frac{d^2G}{dt^2} + m\eta\frac{dG}{dt} + \underbrace{m\omega_0^2 G}_{\text{small } O(\varepsilon)} \right] = \int_{t_0-\varepsilon}^{t_0+\varepsilon} dt\, \delta(t-t_0)$$

Yielding

$$\overset{0 \text{ see picture this is } O(\varepsilon)}{\quad}$$

$$m \left.\frac{dG}{dt}\right|_{t_0-\varepsilon}^{t_0+\varepsilon} + \cancel{m\eta\, G\Big|_{t_0-\varepsilon}^{t_0+\varepsilon}} = 1$$

Using that $G(t,t_0)$ vanishes for $t < t_0$, and thus for t slightly greater than t0

$$\star \qquad \boxed{\lim_{t \to t_0^+} G(t,t_0) = 0} \qquad \longleftarrow \text{ follows by Continuity (see picture)}$$

we find

$$\star \qquad \boxed{\lim_{t \to t_0^+} m \frac{dG(t,t_0)}{dt} = 1} \qquad \longleftarrow \text{ Gives slope after kick}$$

Now for $t > t_0$, $G(t,t_0)$ is a solution to the homogeneous equation. The general solution is

$$G(t,t_0) = A\, e^{-\eta/2\, t} \cos(\omega t + \phi) \qquad t > t_0$$

Adjusting the constants, A, ϕ, to reproduce the boundary conditions ($\star$) gives:

$$G(t,t_0) = \frac{\Theta(t-t_0)\, e^{-\eta/2\,(t-t_0)} \sin(\omega_0 (t-t_0))}{m\omega_0}$$

Green function for damped SHO!

Green Fcn for Transients: An example

- Consider a force f_0 which turns on at $t = 0$:

$$f(t) = \begin{cases} 0 & t < t_0 \\ f_0 & t > t_0 \end{cases}$$

- Then

$$x_s(t) = \int_{-\infty}^{\infty} \overbrace{G_R(t, t_0)}^{\text{vanishes for } t_0 > t} \underbrace{f(t_0)}_{\text{vanishes for } t_0 < 0} \, dt_0$$

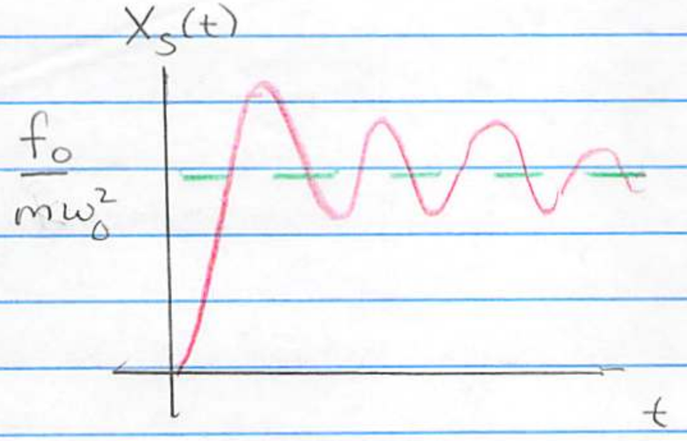
$$= \int_0^t f_0 \, e^{-\eta/2 (t - t_0)} \frac{\sin(\omega_0 (t - t_0))}{m\omega_0} \, dt_0$$

- It is easy to do this integral, writing $\sin \omega t = \mathrm{Im}\, e^{i\omega t}$:

$$x_s(t) = \mathrm{Im} \int_0^t \frac{f_0}{m\omega_0} e^{-\eta/2 (t - t_0)} e^{i\omega_0 (t - t_0)} \, dt_0$$

$$\simeq \frac{f_0}{m\omega_0^2} \left[1 - e^{-(\eta/2)t} \left(\cos(\omega_0 t) + \frac{\eta}{2\omega_0} \sin(\omega_0 t) \right) \right]$$

Where we have neglected terms of order $(\eta/\omega_0)^2 \ll 1$. As we have already done in deriving the Green function.



for a constant force, the equilibrium value is f_0/k with $k = m\omega_0^2$.