## Fourier Method for Green fcn

$$\int \frac{d\omega}{2\pi} e^{-i\omega\tau}$$

$$\left[ m\frac{d^2}{d\tau^2} + m\eta\frac{d}{d\tau} + m\omega_0^2 \right] G_R(\tau) = \delta(\tau)$$

Fourier Transform both sides

$$\left[ -m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right] G_R(\omega) = 1$$

$$G_R(\omega) = \frac{1/m}{\left[-\omega^2 + \omega_0^2 - i\omega\eta\right]}$$

Thus

$$G_R(\tau) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau} \, 1/m}{\left[-\omega^2 + \omega_0^2 - i\omega\eta\right]}$$

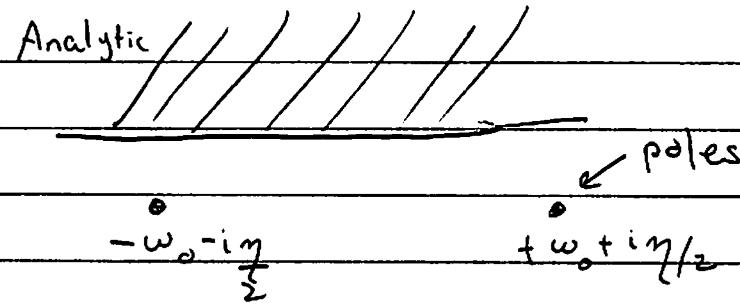
You can do these integrals with contour integration the poles are at

$$\omega^2 + i\omega\eta = \omega_0^2$$

Solving this equation for small $\eta$:

$$\omega \simeq \pm\omega_0 - i\frac{\eta}{2}$$

We see that the integrand has the following analytic structure



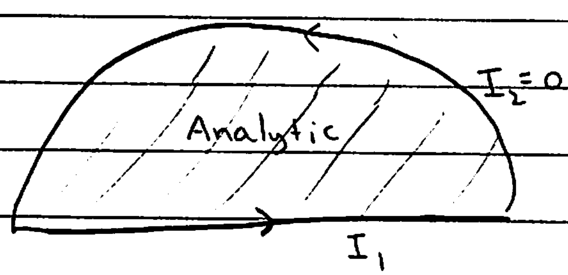

So now we should do the integral:

Case 1: $\tau < 0$ $\quad G_R(\tau) = 0 \leftarrow$ causality

The math works like this, since $\tau < 0$:

$$e^{-i\omega\tau} \xrightarrow[\omega \to \text{complex}]{} e^{-i\,\text{Re}\,\omega\,\tau}\,\underbrace{e^{+[\text{Im}\,\omega]\tau}}_{\substack{\text{decreasing exponentially} \\ \text{for Im}\,\omega > 0}} \quad \tau < 0$$

Thus for $\tau < 0$ we can close the contour in the UHP without picking up poles and find zero




$I_1 + \cancel{I_2} = 0$

$I_1 = 0$

Case 2: $\tau > 0$

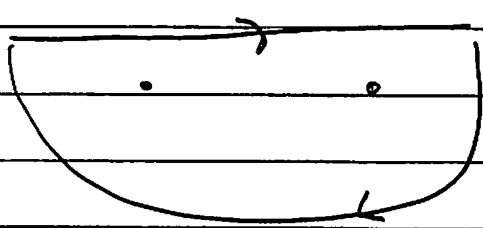
For $\tau > 0$ we must close the contour in the LHP picking up poles at $\omega = \pm\omega_0 - i\frac{\eta}{2}$

For $\tau > 0$: wrong way arand poles

$$G_R(\tau) = -2\pi i \left[ \mathrm{Res}_{\omega = \omega_0 - i\frac{\eta}{2}} + \mathrm{Res}_{\omega = -\omega_0 - i\eta/2} \right]$$

$$= \frac{1}{m}\frac{i}{2\omega_0} e^{-\eta/2\, t} e^{-i\omega_0 \tau} + \frac{1}{m}\frac{-i}{2\omega_0} e^{-\eta/2\, \tau} e^{+i\omega_0 \tau}$$

homogeneous solutions from two pages back



$$= \frac{1}{m} e^{-\eta/2\, \tau} \frac{\sin\omega_0 \tau}{\omega_0}$$

So

$$G_R(\tau) = \Theta(\tau) \frac{\sin\omega_0 \tau}{m\omega_0} e^{-\eta/2\, \tau} \xrightarrow[\eta \to 0]{} \Theta(\tau) \frac{\sin\omega_0 \tau}{m\omega_0}$$

We will see that this Green function is closely related to the green function of the wave eqn