Dynamics and The Euler Equations

- Now we are ready to determine the EOM of the rigid body. The motion of the CM is given by

$$\frac{d\vec{P}_{cm}}{dt} = \vec{F}_{ext}$$

This determines $\vec{P}_{cm}$ and $\vec{V}_{cm}$ and then we can integrate $\vec{v}_{cm}$ to determine $\vec{X}_{cm}$

- Somewhat analogously, the equation of motion for the angles of the rigid body is

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}} \qquad \vec{\tau}_{ext} = \sum_a \vec{r}_a \times \vec{F}_{ext,a}$$

where $\vec{\tau}$ external is the sum of the external torques on the body. This determines $\vec{L}$ and $\vec{\omega}$, and then the angular orientation of the body.

- Let us neglect external forces, to study the free motion of a rigid body

$$\boxed{\left(\frac{d\vec{L}}{dt}\right)_{Lab} = \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega} \times \vec{L}}$$

← describes the change in $\vec{e}_a$

time derivative of $\vec{L}$ assuming the $\vec{e}_a(t)$ are constant

- More explicity (by way of review) for $\vec{L} = L^a \vec{e}_a$

$$\frac{d\vec{L}}{dt} = \left(\frac{dL^a}{dt}\vec{e}_a\right) + \left(L^a \frac{d\vec{e}_a}{dt}\right) \longleftarrow \text{this is } \vec{\omega}\times\vec{L}$$

$$\frac{d\vec{L}}{dt} = \frac{dL^a}{dt}\vec{e}_a + L^a \underline{\hat{\omega}_{ab}}\,\vec{e}_b$$

$$= \frac{dL_a}{dt}\vec{e}_a + \underline{\varepsilon_{cab}}\,\omega^c L^a \vec{e}_b$$

- After shuffling indices, the EOM for $\vec{\tau}_{ext} = 0$

$$\frac{dL_a}{dt} + \varepsilon_{abc}\,\omega^b L^c = (\cancel{\vec{\tau}_{ext}})_a \quad (\star)$$

free motion $\vec{\tau}_{ext} = 0$

Now we will work with the principal axes:

$$L_1 = I_1\omega_1 \qquad L_2 = I_2\omega_2 \qquad L_3 = I_3\omega_3$$

Then writing out Eq $\star$ we find the Euler equations

$$\boxed{\begin{aligned} I_1 \frac{d\omega_1}{dt} &= \omega_2\omega_3(I_2 - I_3) \\ I_2 \frac{d\omega_2}{dt} &= \omega_3\omega_1(I_3 - I_1) \\ I_3 \frac{d\omega_3}{dt} &= \omega_1\omega_2(I_1 - I_2) \end{aligned}}$$

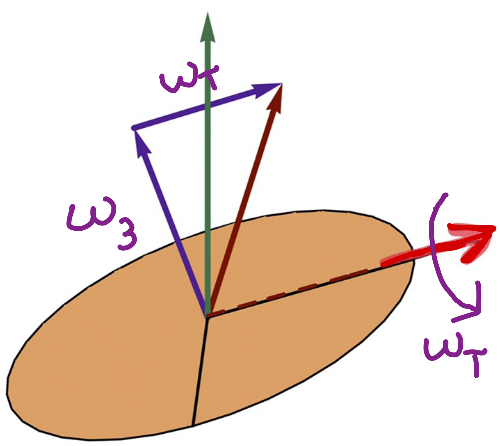
Euler Equations for free motion

Spinning Plate Intro

- First watch the experiment (click me!)
  Notice that the plate wobbles, and the direction of the tilt turns in time

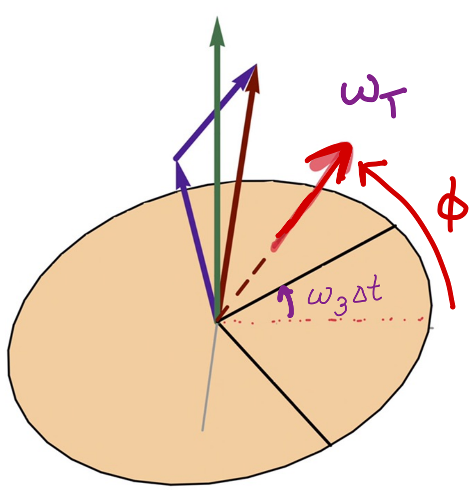
- Next watch the slow motion simulation (click me. And find the .mov file)

$t = 0$



- The red arrow shows the direction of the maximum tilt, and how it turns in time. The rate of turn, is called $\dot{\phi}$, the precession rate.

$t = \Delta t$



- In each time moment the plate spins around the tilt axis with rate ω_T, I called ω_T the wobble rate

- Of course it also spins around its own $\hat{z}$ axis. This is ω_3, the spin rate

The free Symmetric Top

- Take a plate and send it spinning in the air. Inspection shows that in addition to spinning, it wobbles (watch video). We should predict this the wobble rate.

- Euler wrote down his equations to predict the Wobbling of the earth

- We will first study a "Symmetric top" which is a rigid body with moment of inertia's

$$I_1 = I_2 \neq I_3$$

- We will study the spinning plate where

$$I_1 = I_2 = \frac{I_3}{2} = \frac{1}{4} M R^2 \qquad I_3 = \frac{1}{2} M R^2$$

- Then the Euler equations become $(I_2 = I_1)$

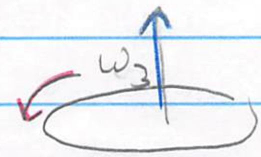
$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_1 - I_3)$$

$$I_1 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$

$$I_3 \dot{\omega}_3 = 0$$

- Examining the equations we see

$$\omega_3 = \text{Const}$$



- And ω_1 and ω_2 satisfy

$$\mathring{\omega}_1 = -\omega_2 \Omega \qquad \text{with} \qquad \boxed{\Omega \equiv \omega_3 \frac{(I_3 - I_1)}{I_1}}$$

$$\mathring{\omega}_2 = +\omega_1 \Omega$$

precession rate

general symm top

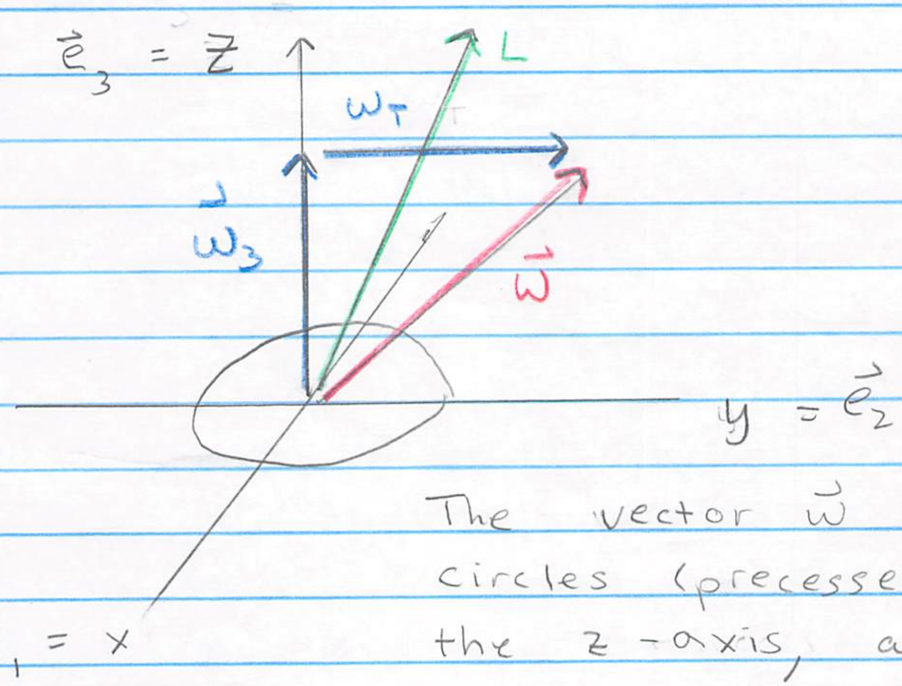
$= +\omega_3$ for disk where $I_3 = 2I_1$

- Now these equations are easily solved:

$$\boxed{\vec{\omega}_T = (\omega_1, \omega_2) = \omega_T(-\sin\Omega t, \cos\Omega t)}$$

precession of $\vec{\omega}_T$

with ω_T a constant (ω-transverse to z')



The vector $\vec{\omega}$ and $\vec{\omega}_T$ circles (precesses) around the z-axis, at a rate $|\Omega| = \omega_3$.

See picture and video (click me)

- The angular momentum lies between $\vec{\omega}$ and $\vec{\omega}_3$, since $I_1 = I_2 = \frac{1}{2} I_3$

$$\vec{L} = I_1 \omega_1 \vec{e}_1 + I_2 \omega_2 \vec{e}_2 + I_3 \omega_3 \vec{e}_3$$

$$= I_1 \underbrace{(\omega_1 \vec{e}_1 + \omega_2 \vec{e}_2)}_{\equiv \vec{\omega}_T} + I_3 \vec{\omega}_3$$

And so:

$$\boxed{\frac{\vec{L}}{I_3} = \frac{1}{2} \vec{\omega}_T + \vec{\omega}_3}$$

$$\vec{\omega} = \vec{\omega}_T + \vec{\omega}_3$$

- Comments

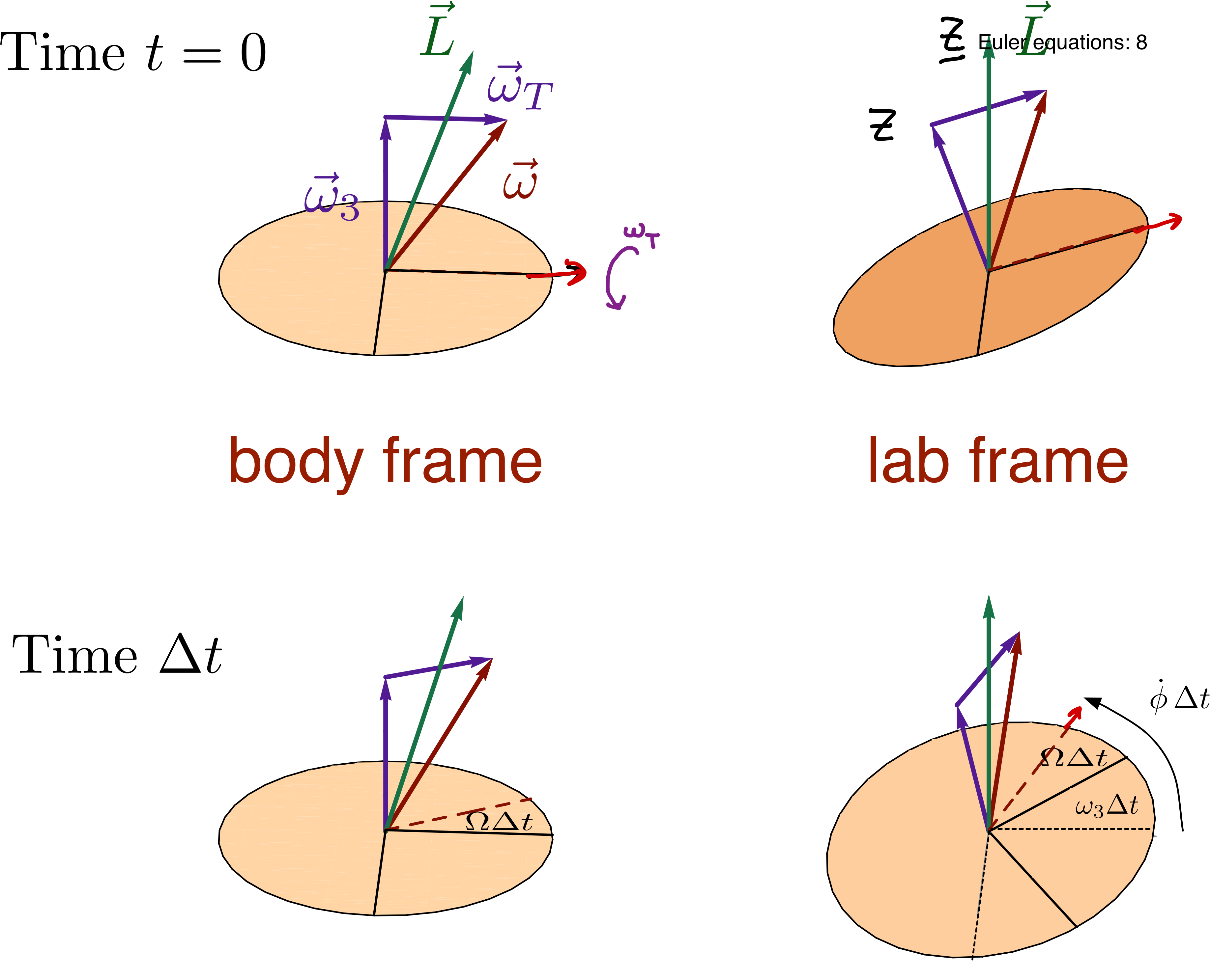
① The fact that $\vec{\omega}_T$ is non zero means the plate will wobble as it spins. ω_T is what we mean by the wobble rate

② In the body frame $\vec{\omega}_T$ precesses with rate Ω. In the Lab frame $\vec{L}$ is constant and can be taken as the $\underline{z}$ axis

③ We will derive it formally next, but intuitively in each time moment Δt the disk rotates by $\omega_3 \Delta t$, and the tilt angle (the direction of $\vec{\omega}_T$) advances by $\Omega \Delta t$ relative to this so the tilt has advanced by $\Delta\phi = (\omega_3 + \Omega)\Delta t$, i.e.

$$\dot{\phi} \simeq \omega_3 + \Omega = 2\omega_3 \quad \text{for disk}$$

See picture! (next page)



Euler equations: 8
Time $t = 0$
$\vec{L}$
$\vec{\omega}_T$
$\vec{\omega}_3$
$\vec{\omega}$
ω_T
z
z
$\vec{L}$
body frame
lab frame
Time Δt
$\Omega \Delta t$
$\dot{\phi}\,\Delta t$
$\Omega \Delta t$
$\omega_3 \Delta t$

Euler Angles of Spinning Plate

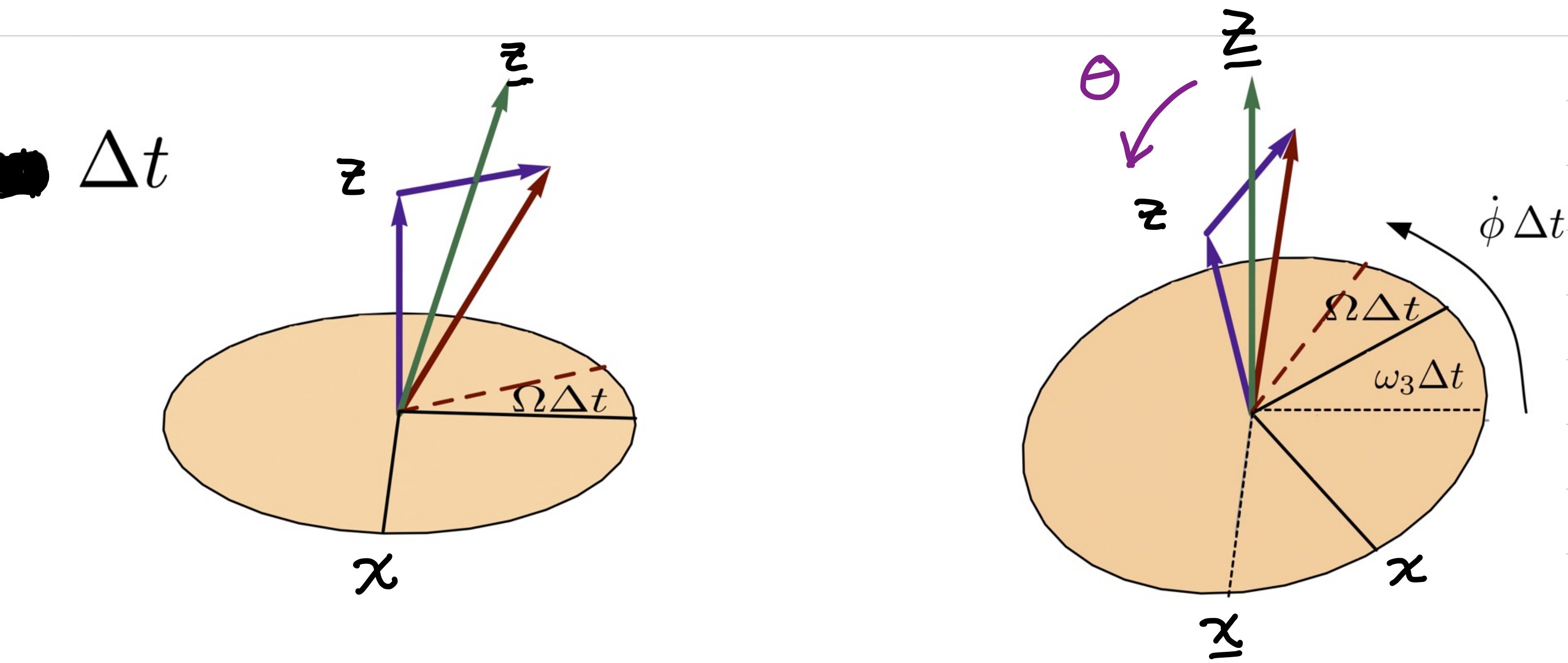
- We have worked out the angular velocity ω_x, ω_y, ω_z now we should relate these to the Euler angles θ, ϕ, ψ.

- Since $\vec{L}$ is constant we orient it along the fixed $\underline{z}$ axis. Then the $\vec{e}_3$ principal axis (the axle of the plate) is directed along $\underline{z}$

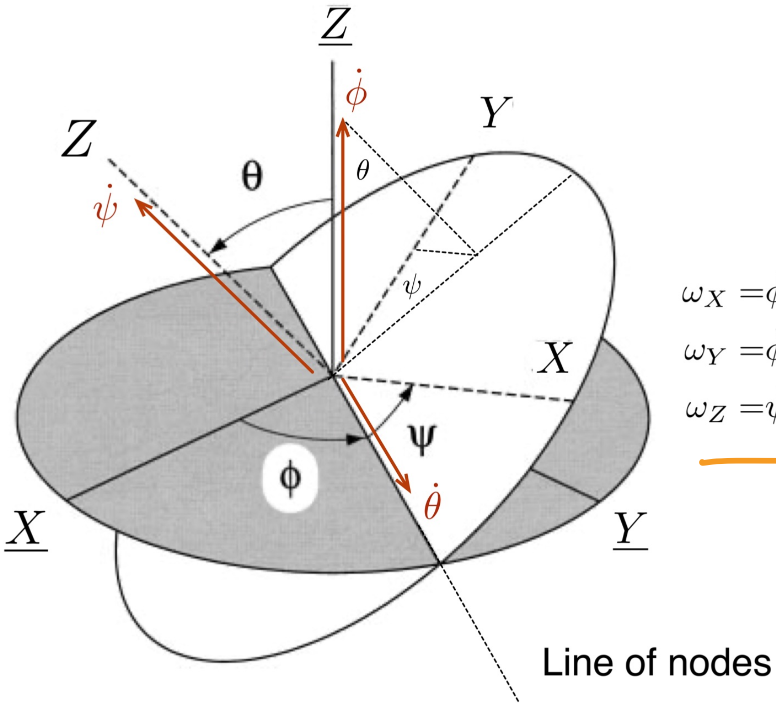
- The angle between $\vec{e}_3$ and $\vec{e}_3$ (z and $\underline{z}$) is θ and this is constant in time

$$\vec{\omega}_3 \cdot \vec{L} = \omega_3 L \cos\theta = \text{const}$$

So $\boxed{\dot{\theta} = 0}$



- Now recall the relation between the Euler angles and $\vec{\omega}$



$\dot\theta = 0!$

$$\begin{aligned}
\omega_X &= \dot\phi\sin(\theta)\sin\psi + \cancelto{0}{\dot\theta\cos(\psi)} \\
\omega_Y &= \dot\phi\sin(\theta)\cos\psi - \cancelto{0}{\dot\theta\sin(\psi)} \\
\omega_Z &= \dot\psi + \dot\phi\cos(\theta)
\end{aligned}$$

- Now here we found

$$\omega_x = -\omega_T \sin\Omega t$$

$$\omega_y = \omega_T \cos\Omega t$$

compare

$$\omega_3 = \omega_z = \text{const}$$

- So comparison gives :

$\omega_T = \dot\phi \sin\theta$ and $\psi = -\Omega t$

i.e. $\dot\psi = -\Omega$

- So comparison shows

$$\boxed{\psi = -\Omega t} \qquad \omega_T = \dot{\phi}\sin\theta$$

Or $\dot{\psi} = -\Omega$.

- Then from the third angular velocity equation

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

We find since $\dot{\psi} = -\Omega$

$$\boxed{\dot{\phi} = \frac{\omega_3 + \Omega}{\cos\theta}}$$

precession ~~wobble~~ rate in terms of the spinning, ω_3, and precession Ω in body frame

- For the plate the tilt is usually small and then, $\cos\theta \simeq 1$. Thus we find:

$$\dot{\phi} \simeq \omega_3 + |\Omega|,$$

as we anticipated by watching the movie. For a disk $\Omega = +\omega_3$ and thus

$$\boxed{\dot{\phi} \simeq 2\omega_3}$$

i.e. the rate of precession $\dot{\phi}$ is approximately twice the rate of spinning, ω_3, as anticipated