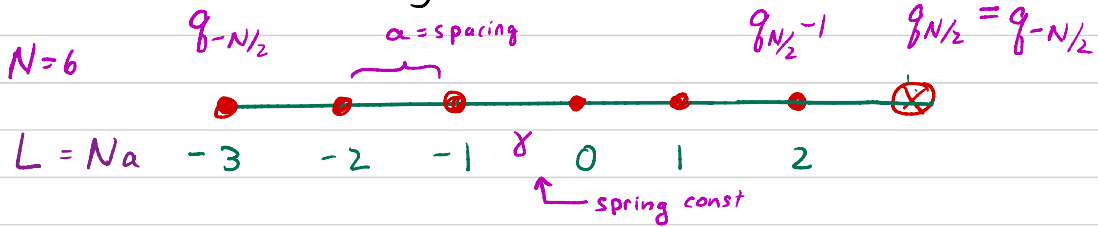


Wave propagation

- Lets return to the discrete case and discuss boundary conditions



- Consider a finite chain with periodic B.C.
 q_j with $j = -N/2, \dots, N/2-1$

★ $q_{N/2} = q_{-N/2}$

- Then exactly as before we would look for eigen solutions. (We will not use the summation convention in this section)

$$q_j = A_k E_j^k e^{-i\omega t} \quad (\text{no sum})$$

- Then as before we find eigen-vectors & values

$$E_j^k = e^{ik \times j} \quad \omega = \pm \omega(k)$$

Now however we must have $q_{N/2} = q_{-N/2}$
or

$$kNa = 2\pi m \Rightarrow k_m = \frac{2\pi m}{L}$$

\uparrow integer

- However not all m lead to independent solution since ($L = Na$, $x_j = ja$, $k_m = 2\pi m/L$)

$$e^{ik_m x_j} = e^{i2\pi j m / N}$$

Thus, $m + N$ yields the same solution as m , and we can take the set

$$k_m = \frac{2\pi m}{L} \quad m = \underbrace{-\frac{N}{2} \dots \frac{N-1}{2}}_{N \text{ modes}}$$

- We have found N eigen modes and we can use this as a basis for the complete solution:

$$q_j = \frac{1}{L} \sum_m A_m e^{ik_m x_j - i\omega(k_m)t} + B_m e^{ik_m x_j + i\omega(k_m)t}$$

↑
inserted for convenience

If q_j is real the term with $-k_m$ in the B-terms, must match the term with k_m in the A-terms leading to $B_{-m} = A_m^*$, i.e.

$$q_j = \frac{1}{L} \sum_m A_m e^{ik_m x_j - i\omega(k)t} + \text{c.c.}$$

↑
complex conjugate of first term

- Finally Lets take the limit $N \rightarrow \infty$
and $a \rightarrow 0$

$$\sum_m \rightarrow \int dm = \int \frac{L dk}{2\pi}$$

$$A_m \rightarrow A(k)$$

Leading to a fourier integral representation :

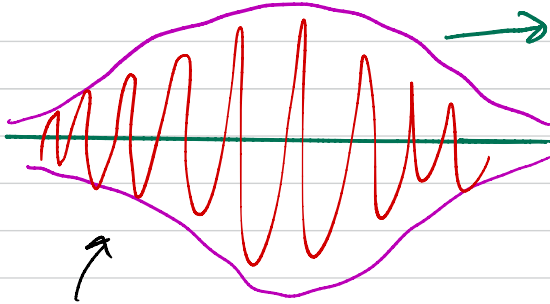
$$q(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega_k t} + c.c.$$

- Lets write out c.c.

$$q(t, x) = \text{Re} \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t} \right]$$

Propagation of Wave Packets

- Now let us consider the propagation of a superposition of waves



typical form

- The wave solution takes the form

$$q(x,t) = \int \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t}$$

we will take complex waves here. In general one would insert "real part" everywhere but that would lead to unnecessary complications, with no insight.

- The wave form at $t=0$ determines the coefficients $A(k)$

$$q(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx} \Rightarrow A(k) = \int_{-\infty}^{\infty} dx u(x,0) e^{-ik \cdot x}$$

- These are Fourier transforms, let's recall some properties of Fourier transforms. (A tutorial type problem is optional on the next homework)

$$\hat{f}(k) \equiv \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \quad \text{Definition}$$

$$f(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \quad \text{Inverse}$$

- Then DO THE HOMEWORK

Space

k-Space

gaussian
 $\sigma \rightarrow 1/\sigma$

$$G(x) = e^{-x^2/2\sigma^2} \iff \hat{G}(k) = \sqrt{2\pi\sigma^2} e^{-k^2\sigma^2/2}$$

phase makes translation

$$f(x) = e^{ik_0x} g(x) \iff \hat{f}(k) = \hat{g}(k-k_0)$$

$$f(x) = g(x-x_0) \iff \hat{f}(k) = e^{-ikx_0} \hat{g}(k)$$

- The uncertainty principle states that if

$$(\Delta x)^2 \equiv \int dx |f(x)|^2 (x - \bar{x})^2$$

$$(\Delta k)^2 \equiv \int \frac{dk}{2\pi} |\hat{f}(k)|^2 (k - \bar{k})^2, \text{ find}$$

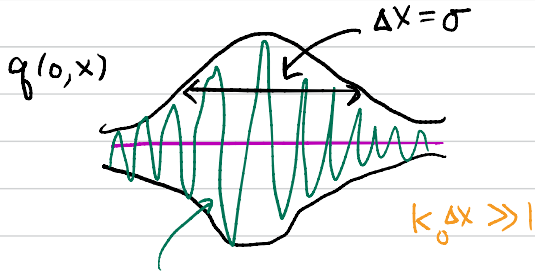
$\Delta k \Delta x \geq 1/2$, with equality holding uniquely for Gaussian.

- The typical wave packet

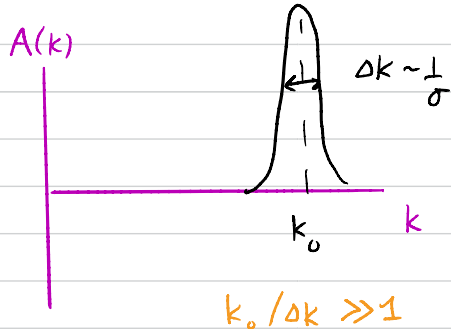
$$q(x, 0) = G(x) e^{ik_0 x} \quad G(x) \equiv e^{-x^2/2\sigma^2}$$

↑ Gaussian

- Then use table



Wavelength determines k_0



- We have $A(k) = \sqrt{2\pi\sigma^2} e^{-\frac{1}{2}(k-k_0)^2\sigma^2}$

- Now we have at future times

$$q(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t}$$

Now we can expand near k_0 , since $A(k)$ is narrow

$$\omega(k) \approx \omega_0 + U(k-k_0)$$

↑
 $\omega(k_0)$

↑
 $U = \left. \frac{d\omega}{dk} \right|_{k_0} \equiv \text{group velocity}$

So:

So

$$q(x,t) = \underbrace{e^{i(Uk_0 - \omega_0)t}}_{\text{phase}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x - Ut)} A(k)$$

- Thus since $A(k)$ was the FTrans of $q(x,0)$

$$q(x,t) = e^{i(Uk_0 - \omega_0)t} q(x - Ut, 0)$$

Since $q(x,0) = G(x) e^{ik_0 x}$ we have

There is a cancellation here: $e^{i(Uk_0 - \omega_0)t} e^{ik_0(x - Ut)} = e^{ik_0 x - \omega_0 t}$

$$q(x,t) = e^{i(k_0 x - \omega_0 t)} G(x - Ut)$$

- So we see that the envelope $G(x)$ of the wave form is shifted by Ut with

$$U = \left. \frac{d\omega}{dk} \right|_{k_0}$$

← this is the speed of the packet

↑ group velocity

- Notice that the trajectory follows the points of stationary phase:

$$\frac{\partial}{\partial k} (kx - \omega(k)t) = 0 \Rightarrow x - Ut = 0$$

The variational problem and dispersion

- Let's recall for an infinite chain of springs, found the dispersion curve:

$$\omega(k) = \pm 2\omega_0 \sin\left(\frac{ka}{2}\right) \quad \omega_0 = \sqrt{\frac{\gamma}{m}}$$

with

$$v_0 = \omega_0 a$$

$$\approx \pm v_0 k \left(1 - \frac{(ka)^2}{24} + \dots\right)$$

- Let's see how the continuous theory can reproduce this result. Consider the action

$$S = \int dt dx \left(\frac{1}{2} \mu (\partial_t q)^2 - \frac{1}{2} Y (\partial_x q)^2 + C (\partial_x^2 q)^2 \right)$$

Then we vary the action, e.g.

$$\begin{aligned} (\partial_x^2 q)^2 &\longrightarrow (\partial_x^2 q + \partial_x^2 \delta q)^2 \\ &\approx (\partial_x^2 q)^2 + 2 (\partial_x^2 q) \partial_x^2 \delta q \end{aligned}$$

• So

$$\delta S = \int dt dx \left(\mu \partial_t q \partial_t \delta q - Y \partial_x q \partial_x \delta q + 2C (\partial_x^2 q) \partial_x^2 \delta q \right)$$

↑
integrate by parts ignoring all boundary terms
2x integrate by parts

- Find

$$\delta S = \int dt dx \left[-\partial_t (\mu \partial_t q) + \partial_x (Y \partial_x q) + 2 \partial_x^2 (C (\partial_x^2 q)) \right] \delta q$$

So the EOM is

$$-\partial_t (\mu \partial_t q) + \partial_x (Y \partial_x q) + 2 \partial_x^2 [C (\partial_x^2 q)] = 0$$

- Treating the parameters as constants we substitute $q = A e^{ikx - i\omega t}$

$$\mu \omega^2 - Y k^2 + 2C k^4 = 0$$

For $C k^4 \ll Y k^2$ we find $\omega = \pm (Y/\mu k^2 - \frac{C}{\mu} k^4)^{1/2}$

$$\omega \approx \pm \sqrt{\frac{Y}{\mu}} k \left(1 - \frac{C}{Y} k^2 \dots \right)$$

- Thus by choosing C and Y

$$\sqrt{\frac{Y}{\mu}} = \omega_0 a \quad \text{and} \quad \frac{C}{Y} = \frac{a^2}{24}$$

we can reproduce the microscopic theory.

★ In general by adding more and more derivatives to the action, such as $(\partial_x^3 q)^2$, even higher terms of the micro-theory can be reproduced by tuning the "low-energy" constants such as C in this case.