Assignment # 1

- Starting from the fact that $\hat{P} = -i\hbar\frac{\partial}{\partial x}$ show that the kinetic energy is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

- Consider the particle in the box as worked out in class. Suppose that at time $t = t_o$ the particle has wave function

$$\psi(x, t_o) = \frac{1}{\sqrt{L}}\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right] \tag{1}$$

  - Write down the eigen functions and eigenvalues of the Hamiltonian for this problem.
  - Express $\psi(x, t_o)$ in terms of the Eigenfunctions of the Hamiltonian, i.e. writing

    $$\psi(x, t_o) = \sum_{E_n} \phi_{E_n}(x)\psi(E_n, t_o) \tag{2}$$

    determine $\psi(E_n, t_o)$
  - Determine the wave function at some later time t.
  - Determine the average position and average momentum as a function of time

    $$\left\langle \hat{X}(t) \right\rangle = \int dx\, x\, \psi^*(x, t)\psi(x, t) \tag{3}$$

    $$\left\langle \hat{P}(t) \right\rangle = \int dx\, \psi^*(x, t) - i\hbar\frac{\partial}{\partial x}\psi(x, t) \tag{4}$$

    Make a graph of $\langle X(t)\rangle$ and $\langle P(t)\rangle$.
  - Show that

    $$\partial_t \left\langle \hat{X}(t) \right\rangle = \frac{\left\langle \hat{P}(t) \right\rangle}{M} \tag{5}$$

- Consider the function shown in figure one. Determine its Fourier series expansion. First in terms of sin and cos and then in terms of e^{ikx} If you can find a computer running mathematica and plot the first couple of terms in this series.

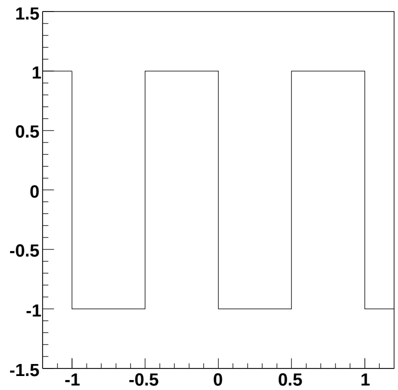

Figure 1: Take the size of the box to be -0.5 to 0.5.

- Determine the Fourier transform $f(k)$ of the following function.

$$f(x) = \frac{1}{2a}\, e^{-\frac{|x|}{a}} \tag{6}$$

  Make a graph of $f(k)$ for large and small a

- The lowest eignenstate of harmonic oscillator is

$$\phi_E(x) = \left(\frac{a^2}{\pi}\right)^{1/4} e^{-\frac{1}{2}(ax)^2} \tag{7}$$

  – Start by proving the following integrals using the tricks on the following page

$$\int_{-\infty}^{\infty} du\, e^{-au^2+bu} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \tag{8}$$

$$\int_{-\infty}^{\infty} du\, u^2\, e^{-au^2} = \sqrt{\frac{\pi}{a}}\, \frac{1}{2a} \tag{9}$$

  Use this integrals to show that

$$\int dx\, |\phi_E(x)|^2 = 1 \tag{10}$$

– Determine the average kinetic energy

$$\int_{-\infty}^{+\infty} dx\, \phi_E^*(x)\, \frac{\hat{P}^2}{2M}\, \phi_E(x) \tag{11}$$

by taking derivatives and performing the integral over x.

– Determine the wave function in momentum space $\phi_E(p)$ where

$$\phi_E(x) = \sum_p \frac{e^{+ipx}}{\sqrt{L}} \phi_E(p) \tag{12}$$

Make a graph of $\phi_E(p)$ for large and small a. $|\phi_E(p)|^2\, dp$ can be interperted as the probability that the wave function as momentum between p and $p + dp$

– Starting with Eq. 11 and Eq. 12 show that the average kinetic energy is

$$\sum_p |\phi_E(p)|^2\, \frac{p^2}{2M} \tag{13}$$

Take $\phi_E(p)$ from the previous problem, convert the sum over p to an integral (as discussed in class), and perform the integration. The answer should be the same as before

Gaussian integrals are handled as follows. Consider the following integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \tag{14}$$

Then

$$I^2 = \int dx\, e^{-x^2} \int dy\, e^{-y^2} = \int \int dx dy e^{-x^2+y^2} \tag{15}$$

Then we can change to polar coordinates to write this as

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} r dr e^{-r^2} \tag{16}$$

$$= (2\pi)\,\frac{1}{2} = \pi \tag{17}$$

Thus we have

$$I = \sqrt{\pi} \tag{18}$$

Next consider a slight generaliztion of this

$$I(j) = \int_{-\infty}^{\infty} dx\, e^{-x^2+jx} \tag{19}$$

We may complete the square, $-x^2 + Jx = -(x - j/2)^2 + (j/2)^2$, and then shift the integration variables $y \equiv (x - j/2)$

$$I(j) = \int_{-\infty}^{\infty} e^{-x^2+jx} = \int dy e^{-y^2+(j/2)^2} = e^{(j/2)^2}\,\sqrt{\pi} \tag{20}$$

This allows us to compute integrals of polynomial time gaussian. For example

$$\int_{-\infty}^{\infty} dx\, x^2\, e^{-x^2} = \left[\frac{\partial}{\partial j}\,\frac{\partial}{\partial j} \int dx\, e^{-x^2+jx}\right]_{j=0} = \left[\frac{\partial}{\partial j}\,\frac{\partial}{\partial j} e^{(j/2)^2}\,\sqrt{\pi}\right]_{j=0} \tag{21}$$

$$= \frac{1}{2}\,\sqrt{\pi} \tag{22}$$