Assignment # 2

- Consider a fermi gas of spin 1/2 particles. Compute the ratio between the average momentum and the fermi momentum of these particles in one, two, and three dimensions.

- Starting with the eigen-energies for a particle in a box

$$E_k = \frac{\hbar^2}{2m}\left(\underbrace{k_x^2}_{\frac{\pi n_x}{L}} + k_y^2 + k_z^2\right) \qquad n_x, n_y, n_z = 1, 2, 3, \ldots \tag{1}$$

  The number of states between E and $E + dE$ with say spin up is known as the density of states $\rho(E)dE$. Show that

$$\rho(E)dE = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{E}\,`dE \tag{2}$$

- The free particle eigen functions with with periodic boundary conditions are

$$\phi_E(\mathbf{r}) = \frac{1}{\sqrt{V}}\, e^{+i\mathbf{k}\cdot\mathbf{r}} \qquad k_x = n_x \frac{2\pi}{L} \text{ with } n_x = \ldots, -2, -1, 0, 1, 2, \ldots \tag{3}$$

  with $V = L_x L_y L_z$ and we have written only the conditions on k_x. Similar conditions apply for k_y and k_z Periodic boundary conditions mean that

$$\phi_E(x + L_x, y, z) = \phi_E(x, y + L_y, z) = \phi_E(x, y, z + L_z) = \phi_E(x, y, z)$$

  The eigen energies for this wave function are

$$E_k = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right) \tag{4}$$

  - Show that these wave functions satisfy the periodic boundary conditions.
  - Show that the density of states $\rho(E)dE$ is the same as in the last problem. (Hint. Now the integrals over is over the full sphere instead of just 1/8 of the sphere as in the last problem.)

- Identical particles fill all states up to the fermi energy. Thus, if N is the number of identical particles in a volume V

$$N = \int_0^{\epsilon_F} \rho(E) dE \tag{5}$$

determines the fermi energy. Perform this integral and show that the result is consistent with the result found in class

$$\left(6\pi^2 \frac{N}{V}\right)^{1/3} = k_F \tag{6}$$

where N/V is the density of say spin-up identical particles.

- When a fermi gas expands it does work

$$dW = pdV \tag{7}$$

The total number of particles is fixed in this expansion. The change in energy is

$$dE = -\, dW = -pdV \tag{8}$$

and thus

$$p = -\left(\frac{\partial E}{\partial V}\right)_N \tag{9}$$

Compute the pressure of a fermi gas.

- Estimate the Fermi energy in Copper.
  - Copper has a density of $8.96\,\mathrm{g/cm}^3$. It has one "free electron" in its outer shell which can be either spin up or spin down. Cu has 29 protons and 35 neutrons. Use the fact that 1 Avagadro number of nucleons weighs approximately one gram to find that the density of free spin up electrons is approximately

    $$\frac{N}{V} = 0.042 \frac{1}{\mathring{A}^3} \tag{10}$$

    where $1\mathring{A} = 10^{-10}$m
  - What is the approximate spacing between these electrons and how does this compare to the Bohr radius $a_0 \approx 0.5\mathring{A}$.

- Evaluate the fermi energy of Cu. You should find it to be about 7eV Approximately, how does this compare with the $k_B T$ at room temperature.

- A particle in a fermi gas does not always have the same momentum. A easure of the fluctuations in the momentum of a typical particle is

$$p_{\rm rms} \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \tag{11}$$

  - Compute $\langle p^2 \rangle$ and use the results of problem 1 for $\langle p \rangle$ to determine

$$\frac{p_{\rm rms}}{\langle p_F \rangle} \tag{12}$$

    i.e. the spread of momenta for the particles in the fermi gas