Contents

1 Circuits: Chapter 29

1.1 Kirchoff rules and circuit analysis

(a) Kirchoff rules state that:

i) The total current entering a junction equals the current exiting a junction

$$\text{current in} = \text{current out}$$

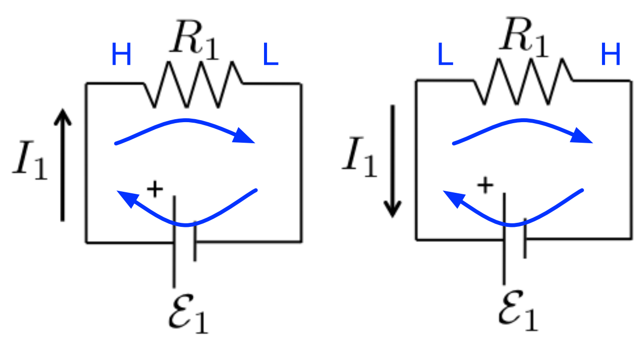
ii) The total change in potential around a closed loop is zero

$$\sum \Delta V = 0$$

(b) To analyze a circuit we first make a reasonable guess for the current direction, then march around the circuit calculating the voltage drop across each circuit element. If your guess was wrong the current will come out negative.

The voltage <u>*drop*</u> for a resistor is IR, $\Delta V = -IR$. The voltage <u>*drop*</u> for a capacitor is Q/C, $\Delta V = -Q/C$. Let us analyze the simplest circuit in four ways. We will get the same answer in all four ways.

Consider the first two cases of the same circuit shown below.



i) In the first case we have taken the current in the most natural way. Current flows from high potential (H) to low potential (L). Marching around the circuit in the direction of the blue arrows, Kirchoff rules say

$$+\mathcal{E} - IR = 0 \qquad \text{thus} \qquad I = \mathcal{E}/R \tag{1.1}$$

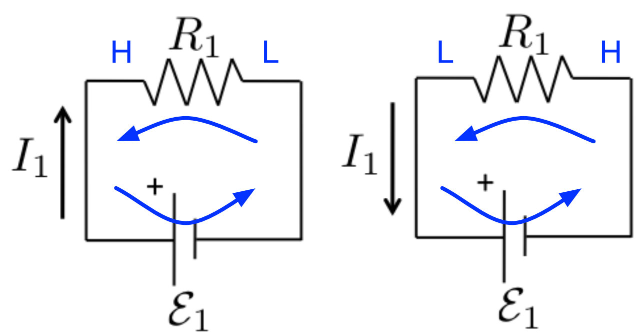
Here $+\mathcal{E}$ is the voltage change across the battery marching in the direction of the blue arrows.

ii) In the second case we have assumed that the current flows in a counter-clockwise direction. Since current flows from high to low potential (labelled H and L in the picture), the voltage change across the resistor in the direction of the arrows is $+IR$

$$+\mathcal{E} + IR = 0 \qquad \text{thus} \qquad I = -\mathcal{E}/R \tag{1.2}$$

Thus we see that the current in the counter-clockwise direction is negative, i.e. it flows in the clockwise direction (as before).

iii) The direction one gows around the loop is arbitrary. Consider the same circuit but analyze it by going counterclockwise around the loop. We will get the same answer. In the first panel we going



counterclockwise around the loop, and we have

$$-\mathcal{E} + IR = 0 \qquad \text{thus} \qquad I = \mathcal{E}/R \tag{1.3}$$

The change in potenial across the battery now is $-\mathcal{E}$ since we are going against the battery. The potential change across the resistor is from low (L) to high (H) or $\Delta V = +IR$.

iv) Finally in the last case (going counter clockwise around) with counter clockwise current (right figure), we have

$$-\mathcal{E} - IR = 0 \qquad \text{thus} \qquad I = -\mathcal{E}/R \tag{1.4}$$

Thus the counterclockwise current is negative, i.e. the current is clockwise.

(c) (taught by Dr. Fernandez) The work done per time by the source (usually a battery) to maintain a potential drop across a resistor or a capacitor is

$$P = IV\,, \tag{1.5}$$

where V is the potential drop across the resistor or capacitor. For a resistor $V = IR$, for a capacitor $V = Q/C$. You should understand where this formula comes from (see Sec. 2.2).

(d) When two or more resistors are in series the "equivalent" resistance is the sum of the resistors.

$$R_{\text{equiv}} = R_1 + R_2 + R_3$$

The situation is shown below

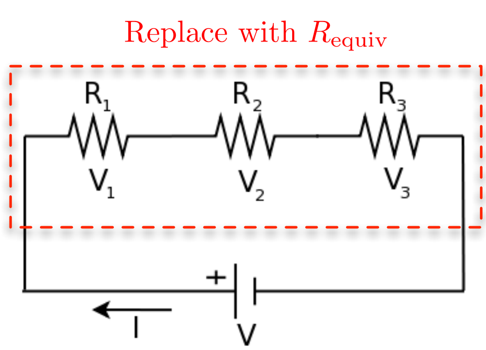


Figure 1.1: Resistors in series

(e) When two or more resistors are in parallel the "equivalent" resistance is found by adding the resistances in reciprocal

$$\frac{1}{R_{\rm equiv}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{1.6}$$

The situation is shown below

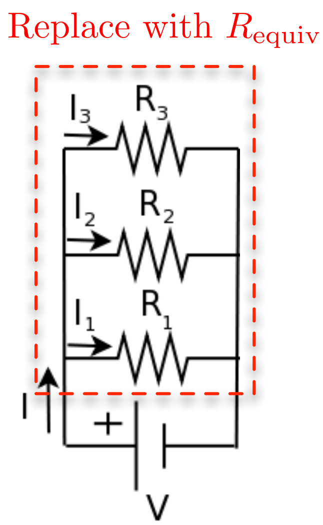


Figure 1.2: Resistors in parallel

1.2 RC circuits

(a) Consider a circuit with a resistor and a capacitor (an "*RC*" circuit). When the switch is closed the battery will push charge on the capacitor. The charge will build up on the capacitor with time until the capacitor's voltage on the top plate exactly matches the voltage of the battery.

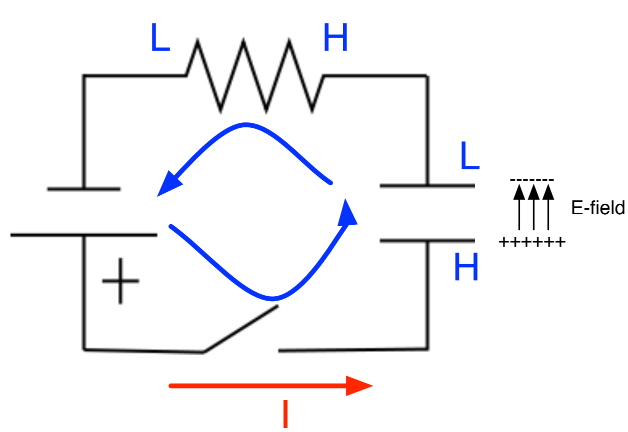


Figure 1.3: An *RC* circuit. The red arrow indicates the direction of the current. The blue arrows indicate how we are marching around the loop in Eq. (1.8). The H and L indicate which side of resistor or capacitor is at higher and lower potential.

(b) The voltage change across the capacitor as we move from High to Low (H to L) is

$$\Delta V = -\frac{Q(t)}{C} \tag{1.7}$$

If we have positive charge $Q(t)$ on the bottom plate as (shown) then the voltage change is negative as we move from the bottom plate to the top plate (High to Low), i.e. the voltage *drop* is $Q(t)/C$. Usually it is the voltage drop that is quoted.

(c) Kirchoff's Law for the RC circuit reads

$$\mathcal{E} - \frac{Q(t)}{C} - I(t)R = 0 \tag{1.8}$$

You should feel comfortable deriving this result.

(d) Eq. (1.8) can be solved for the current as a function of time

$$I(t) = I_o e^{-t/\tau} \tag{1.9}$$

where $I_o \equiv \mathcal{E}/R$ is the initial current just after the switch is closed, and $\tau = RC$ is the time constant of the circuit. While you are not required to derive this result you should be able to do the following:

i) Explain qualitatively why the current is decreasing and why the voltage drop across the capacitor is increasing.

ii) Explain qualitatively and using Eq. (1.8) why the initial current I_o is $\mathcal{E}/R$.

iii) Show (using Eq. (1.8) and Eq. (1.9)) that the potential across the capacitor as a function of time increases as

$$V(t) = \mathcal{E}(1 - e^{-t/\tau}) \tag{1.10}$$

2 Forces on currents in a magnetic fields: Chapter 30

2.1 Force on a current carrying wire in a magnetic field

(a) A current carrying wire of length ℓ in a constant magnetic field $\vec{\boldsymbol{B}}$ experiences a force. The force is

$$\vec{\boldsymbol{F}} = I\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{B}} \tag{2.1}$$

Here the vector $\vec{\boldsymbol{\ell}}$ is the length of the wire in the magnetic field, and is the pointed along the wire in the direction wire's current flow.

As with all cross products the magnitude of the force is

$$F = I\ell B \sin(\theta) \tag{2.2}$$

where $\sin\theta$ is the angle between the ℓ and B. A schematic is shown below

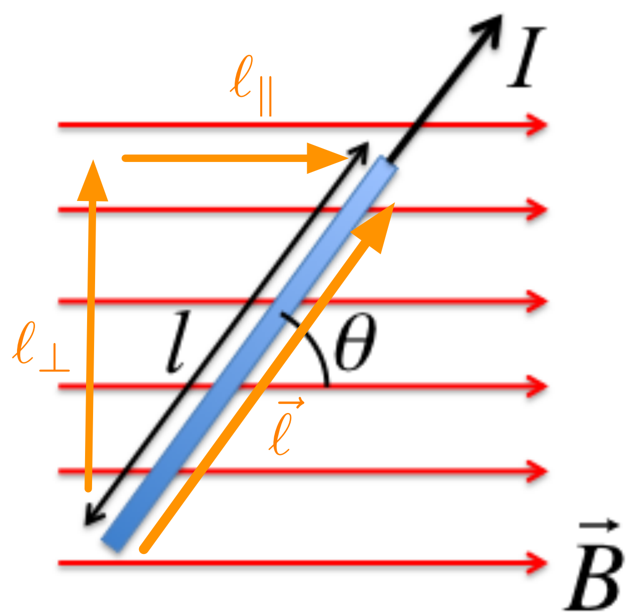


Figure 2.1: Force on a wire

Also note that the magnitude of the component of $\vec{\boldsymbol{\ell}}$ perpendicular to $\vec{\boldsymbol{B}}$, i.e. $\ell_{\perp} = \ell\sin(\theta)$, so the magnitude of the force is

$$F = I\ell_{\perp}B \tag{2.3}$$

The direction of the force is given by the right-hand-rule. In the Fig. 2.1 the force is into the page.

2.2 Force and motion of a charged particle in electric and magnetic field

Electric fields

Prior to my taking over the course you learned the following

(a) In an electric field a particle can accelerate. The change in potential energy of a charge particle is

$$\Delta U = q\Delta V \tag{2.4}$$

where ΔV is the change in voltage of the particle. In particular if a charged particle is accelerated or decelerated through a potential change of ΔV there is a change in kinetic energy. Since $\Delta K + \Delta U = 0$, the change in kinetic energy is

$$\Delta K = -q\Delta V \tag{2.5}$$

(b) If you have a steady current, maintained by a battery driving charge from low to high potential (so the potential change is ΔV), then the work per time or the power delivered by the battery is

$$P = \frac{\Delta U}{\Delta t} = I\Delta V \tag{2.6}$$

(c) When measuring energies of charged particles (such as electrons and protons) it is most convenient to use electron-volts, $1\,eV = 1.6 \times 10^{-19} J$. The reason why it is used is because if a proton (or electron) is accelerated through a potential of $2V$, its kinetic energy will be $2\,\mathrm{eV}$. Thus, when doubly ionized helium (which consists of two protons and has a charge of $2 \times$ proton charge) is accelerated through $2V$, its final kinetic energy is $4\,\mathrm{eV}$.

Magnetic fields

(a) A particle moving with velocity v in an *electric* and *magnetic* field experiences a force

$$\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{E}} + \boldsymbol{v} \times \vec{\boldsymbol{B}}) \tag{2.7}$$

(b) The magnetic field does not change the velocity, but only changes the direction of the velocity. This causes a particle to move in a circle. Newton's Law says that the magnetic force balances the centripetal acceleration

$$qvB = m\frac{v^2}{r}\,, \tag{2.8}$$

This determines the radius of the rotational motion r for a given magnetic field and particle velocity. You should understand where Eq. (2.8) comes from, and how it can be used to determine r. You should also be able to predict the direction a proton or electron will circulate in a magnetic field. (Don't forget that electrons have negative sign).

2.3 Torque on a current loop in a magnetic field and the magnetic moment

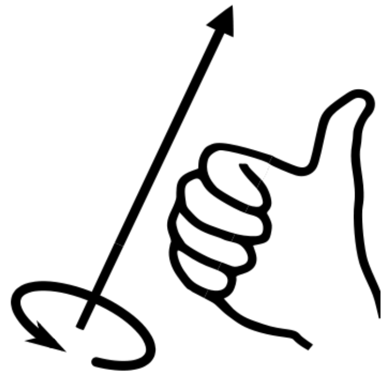
(a) A current loop in a magnetic field experiences a torque and wants to rotate so that the magnetic moment (see below) of the loop points in the direction of the magnetic field. The current loop acts like a compass needle.

(b) The magnetic moment of a current loop is defined

$$\vec{\boldsymbol{\mu}} = NI\,\vec{\boldsymbol{A}} \tag{2.9}$$

where I is the current and $\vec{\boldsymbol{A}}$ is the area vector. N is the number of turns of the current loop.

i) The magnitude of the area vector is just the regular old area (e.g. πr^2 for a circle). The direction of the area vector is normal to the face of the area and is given by the right hand screw rule



In this case one curls ones fingers around the loop (in the direction of the current or circulation of the loop) and your thumb points in the direction of the area or the magnetic moment. The area vector thus describes the orientation of the face of the area.

ii) The magnitude of the torque on a current loop in a constant magnetic field is

$$\tau = \mu B \sin\theta \tag{2.10}$$

where θ is the angle between the magnetic moment and the magnetic field.

3 Currents as a source of magnetic field: Chapter 30 and 31

3.1 Qualitative features

(a) A single wire carrying a current I produces a magnetic field. The field curls around the wire. The direction of how the field circulates around the wire is given again by the right-hand-screw rule. In

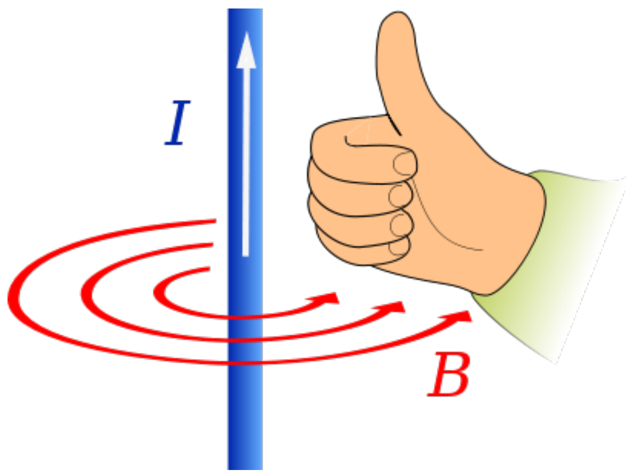


Figure 3.1: right-hand-screw-rule for determining the direction of the Magnetic field

this case one points the thumb in the direction of the current, and the fingers curl in the direction of the field. A formula which we will use a lot is the magnetic field from a long wire (see below)

$$B = \frac{\mu_o I}{2\pi R} \tag{3.1}$$

Here

$$\frac{\mu_o}{4\pi} = 10^{-7}\,\frac{T\cdot m}{A} = 0.1\,\frac{\mu T\cdot m}{A} \tag{3.2}$$

(b) Consider the figure below Fig. 3.2 (described in class). For a current ring shown below you should be able to understand that the current in the picture flows in a counter-clockwise fashion. The magnetic moment of the ring in this picture points down. The field produced by the ring is very similar to a small "dipole" magnet with a north and south pole.

(c) Magnetic fields are measured in Tesla (the SI unit) or Gauss ($1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$). Typical magnetic fields are:

Strong Magnetic Field	$\sim 4\,\mathrm{T}$
Kitchen magnet	$\sim$ mT
Wire carying 1 A	$\sim 1\,\mathrm{G}$
Eaths field	$\sim 0.5\,\mathrm{G}$

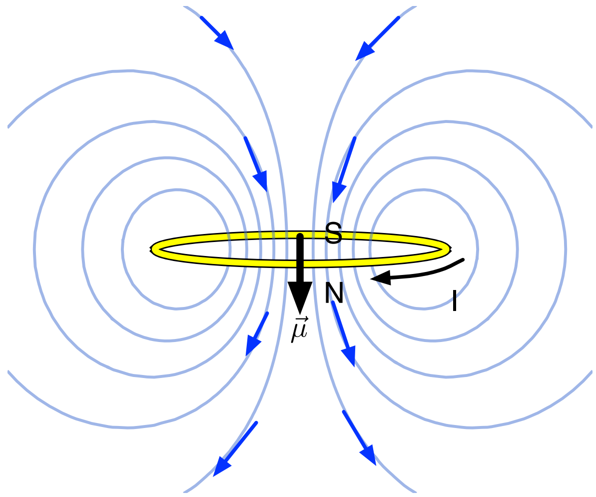


Figure 3.2: Magnetic field of a current ring or dipole magnet

3.2 The magnetic field from currents

(a) Each segment of a wire of length and direction, $d\vec{\boldsymbol{\ell}}$, produces a magnetic field, $d\vec{\boldsymbol{B}}$, according to the Biot-Savat rule shown in Fig. 3.3

$$d\vec{\boldsymbol{B}} = \frac{\mu_o}{4\pi} \frac{I d\vec{\boldsymbol{\ell}} \times \hat{\boldsymbol{r}}}{r^2} \tag{3.3}$$

In this formula, $\hat{\boldsymbol{r}}$ points from the current to the observation point (where you want to compute $\vec{\boldsymbol{B}}$),

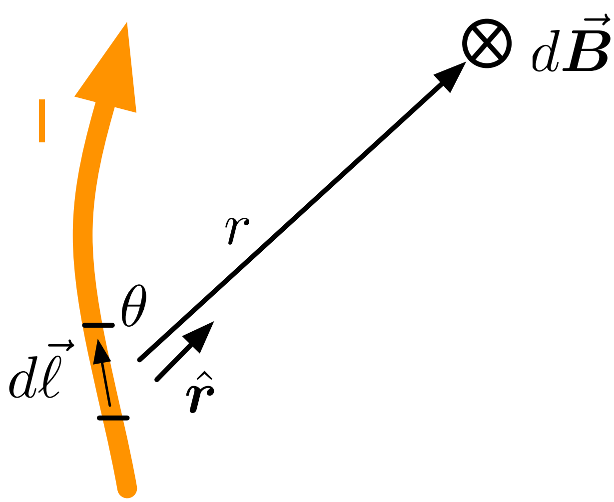


Figure 3.3: Geometry of the Biot Savart Law

and r is the distance between the $d\vec{\boldsymbol{\ell}}$ and the observation point.

As with all cross products, the magnitude of dB is given by

$$dB = \frac{\mu_o}{4\pi} \frac{I d\ell \sin\theta}{r^2} \tag{3.4}$$

where θ is the angle between the two vectors, $d\vec{\ell}$ and $\hat{r}$. Note that $\hat{r}$ is a unit vector and $|\hat{r}| = 1$. The direction is given by the right-hand-rule and points into the page in Fig. 3.3.

(b) The Biot Savart Law and can be used to determine the magnetic field in a number of important cases. In the cases listed below, I have indicated the formula as "derive/remember", depending on whether I expect you to be able to derive or simply remember the result. In all cases you should be able to predict the direction of the field using the right-hand-rule in Eq. (3.3), or the more qualitative

i) (**remember**) The magnetic field of a line of current of length L at the midpoint

$$B = \frac{\mu_o I}{2\pi R} \cos\theta_1 \tag{3.5}$$

Here $\cos\theta_1$ is the angle, and R is the perpendicular distance shown in Fig. 3.4

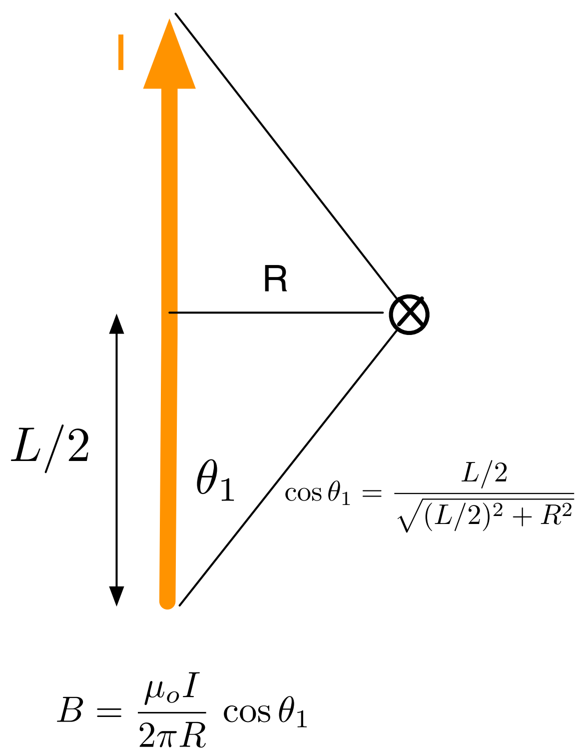


Figure 3.4: The magnetic field from a finite wire

ii) (**derive** using Ampere's Law) The magnetic field of an infinite wire is

$$B = \frac{\mu_o I}{2\pi R} \tag{3.6}$$

This can also be found from Eq. (3.4) by taking $L/2$ very large

iii) **derive** The magnetic field at the center of a ring is

$$B = \frac{\mu_o I}{2R} \tag{3.7}$$

A schematic is shown below in Fig. 3.5 The direction of the field is in the direction of the magnetic moment of the current ring (to the right in fig above). Indeed the magnetic field at the center be written

$$\vec{B} = \frac{\mu_o}{4\pi} \frac{2\vec{\mu}}{R^3} \tag{3.8}$$

where $\mu = I\pi R^2$ is the magnetic moment

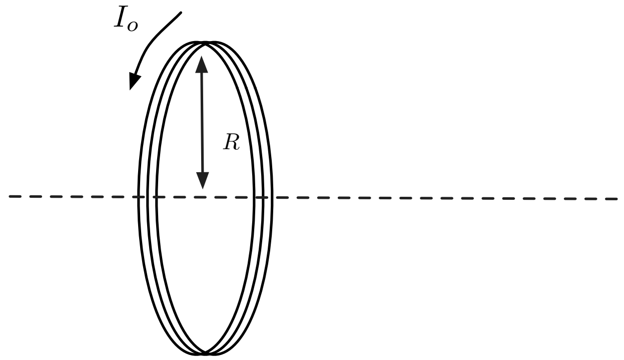


Figure 3.5: Geometry of a current ring

iv) **remember** The field of a ring of current along the axis of a ring. A schematic is shown below in Fig. 3.6 Along the axis of the ring we have

$$\vec{\boldsymbol{B}} = \frac{\mu_o}{4\pi} \frac{2\vec{\boldsymbol{\mu}}}{(R^2 + x^2)^{3/2}} \qquad \mu = I\pi R^2 \tag{3.9}$$

At large distances x, we have

$$\vec{\boldsymbol{B}} = \frac{\mu_o}{4\pi} \frac{2\vec{\boldsymbol{\mu}}}{x^3} \tag{3.10}$$

seeing a characteristic falloff of $1/x^3$. This $1/r^3$ is typical of all magnets which always have a north and south pole. If a north pole existed in isolation (they don't) we would see $1/r^2$ like in the coulomb law.

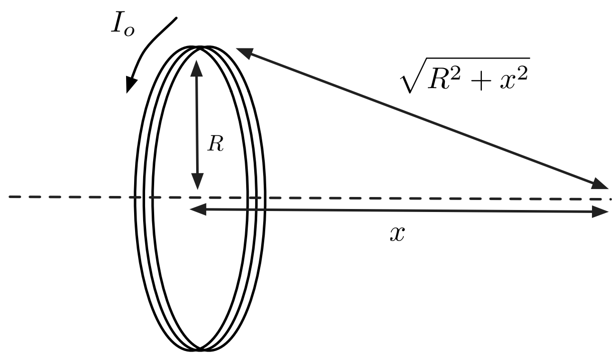


Figure 3.6: Geometry of a current ring on axis but away from center

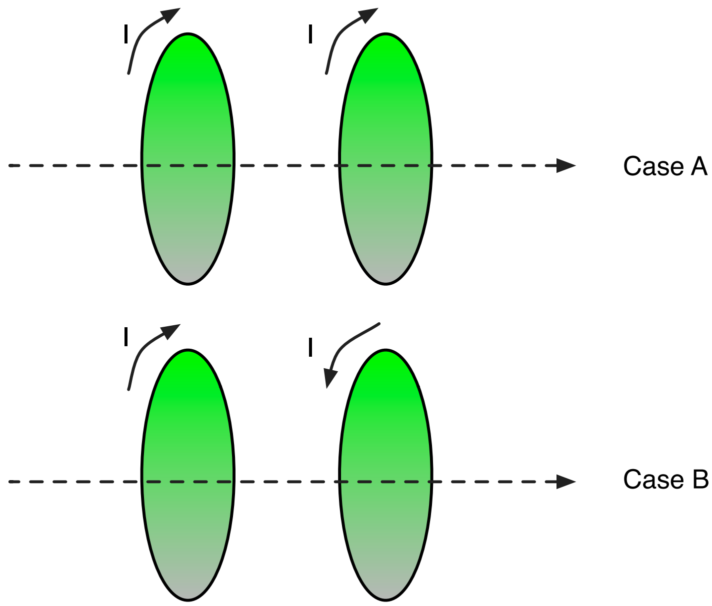
3.3 Forces and torques

(a) We showed that like currents attract and unlike currents repel (i.e. the opposite to charges). You should be feel comfortable deriving that the force per length (F/L) between infinitely long parallel currents is

$$\frac{F}{L} = \frac{\mu_o I^2}{2\pi R} \tag{3.11}$$

and be able to explain why they attract or repel by showing the fields and using $I\ell B \sin\theta$ etc.

(b) For the two rings shown below, the top two rings attract (case a). The bottom two rings repel (case b). However, in the bottom case, the rings will (if they can rotate) eventually turn relative to each other.



Assume that the right ring can rotate. The field from the left ring points to the left. The magnetic moment of the right ring wants to rotate by 180^o in order to align its magnetic moment with the field of the left ring. In Case B the magnetic moment of the left ring points to the left, while the right ring initially points to the right. After the right ring rotates by 180^o, the two rings will start to attract.

3.4 Amperes Law

(a) For any given loop we defined the circulation of the magnetic field about the loop as

$$\text{circulation of } \vec{\boldsymbol{B}} \text{ around a loop} = \oint_{\text{loop}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} \approx \sum_{\text{segments}} \vec{\boldsymbol{B}} \cdot \Delta\vec{\boldsymbol{\ell}} \tag{3.12}$$

Example 31.3 can help you understand the meaning of this integral.

(b) Amperes Law says that the circulation of the magnetic field around an arbitrarily chosen loop equals (up to a constant) the current through that loop

$$\oint_{\text{loop}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_o I_{\text{thru}} \tag{3.13}$$

To understand what is I_{thru} look at Fig. 3.7. We first draw a loop (any loop) including its orientation (in the Fig. 3.7 the orientation is clockwise from above). So the normal, $\boldsymbol{n}$, to the loop points down according to the right hand screw rule. The direction of positive current is given by $\boldsymbol{n}$. In this case the current through the loop is therefore

$$I_{\text{through}} = \underbrace{-2I_o}_{\text{current through from left wire}} + \underbrace{I_o}_{\text{current through from right wire}} \tag{3.14}$$

The first wire is negative because the current direction is opposite to what we called positive (the $\boldsymbol{n}$ direction).

(c) We used Amperes law to derive the field of a solenoid. The field of a solenoid is

$$B = \mu_o n I \tag{3.15}$$

inside the solenoid and zero outside the solenoid. Here $n = N/L$ is the number of turns per length of the solenod. I is the current in the solenoid.

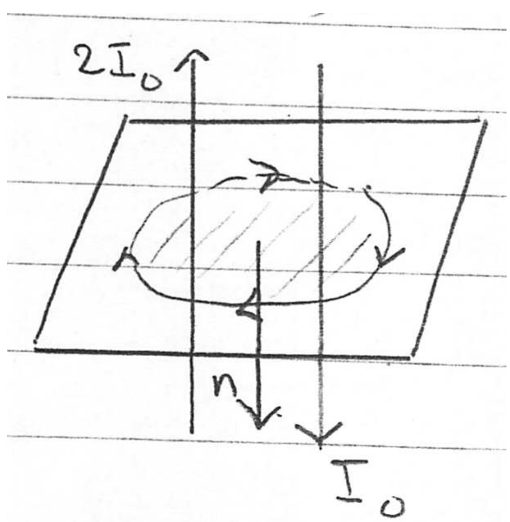


Figure 3.7: Two wires carrying currents $2I_o$ up and I_o down

4 Faraday Law and induction: Chapter 32 and 33

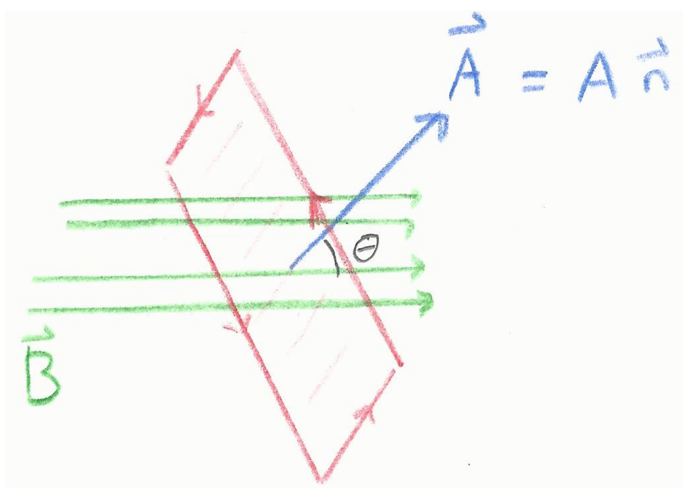
4.1 Magnetic flux, Lenz Law and Faraday Law

Roughly Faraday's Law says that changing electric field produces a $\vec{\boldsymbol{B}}$-field. More precisely it says that the circulation of the electric field (or the in the voltage) is minus the rate in change of the magnetic flux. Below we will summarize the ingrediants.

- The magnetic flux Φ_B is a measure of how much magnetic field passes through a loop. For a loop with area vector $\vec{\boldsymbol{A}} = A\,\boldsymbol{n}$ (i.e. with magnitude A and direction given by the right hand rule), the magnetic flux is

$$\Phi_B \equiv \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}} = BA\cos\theta \tag{4.1}$$

  The flux is positive if $\vec{\boldsymbol{A}}$ (or $\boldsymbol{n}$)and $\vec{\boldsymbol{B}}$ point in the same direction



  This is only for constant $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{A}}$. More generally one would integrate over the surface

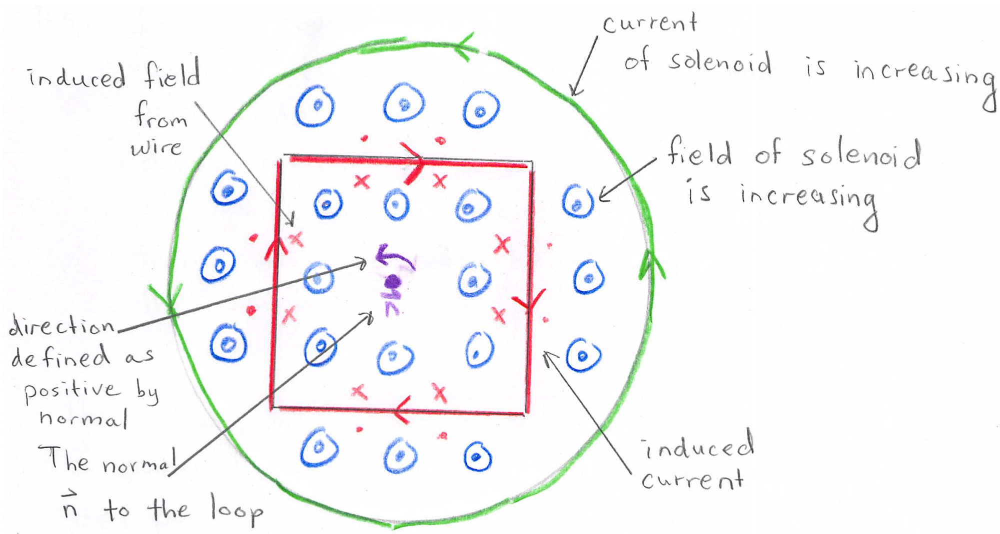
$$\Phi_B \equiv \int_{\text{surface}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} \tag{4.2}$$

  The unit of flux is weber. 1 Weber $= 1Tm^2$

- Changing magnetic fields produce electric fields or voltages. More precisely the emf around a loop $\mathcal{E}$ is minus the change in magnetic flux

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{4.3}$$

  Here N is the number of turns in the loop of wire through which the magnetic field passes.



Here $\mathcal{E}$ is the circulation of the electric field.

$$\mathcal{E} \equiv \oint \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{\ell}} \tag{4.4}$$

- The minus sign in Eq. (4.3) is known as Lenz' law and indicates the direction of the $\mathcal{E}$. It means that the current flows in such a way that the magnetic field which the induced current would produce opposes the change in flux. Consider the example discusse in lecture (Example 32.1 in Book) Draw a loop (the red square) and choose an orientation or direction of circulation – it usually a good idea to take the normal to the loop to point in the same direction as the magnetic field (though you will get the same answer the other way). Thus, in the figure the normal points out of the page, and positive is defined counter clockwise.

  The magnetic field (the blue circles) is increasing and thus the magnetic flux (i.e. the number and strength of the the blue circles passing through the square) is increasing. We will conclude that the red current flows clockwise in two ways:

  (a) **Physics way:** The red current flows clockwise in this case since the secondary induced field from the (red) current tends to make a magnetic field (the red crosses) which then opposes the increasing flux.

  (b) **Math way:** Mathematically the Emf is

  $$\mathcal{E} = -\frac{d\Phi_B}{dt} = -A\frac{dB}{dt} \tag{4.5}$$

  where A is the area of the loop. The Emf is negative since the magnetic field is increasing. Thus the emf is negative and this indicates – since counter-clockwise was defined by the normal as positive – that the emf is clockwise.

4.2 Generating of current flow

We discussed to specific cases of current generation which you should feel comfortable deriving

- We discussed a rod moving on rails (a slide generator Section 32.5). Here the changing flux is caused by the moving rod. The voltage is

$$\mathcal{E} = B\ell v$$

- A generator of AC current by turning a coil of wire in a magnetic field. Here the changing flux is caused by the changing angle $\theta = \omega t$. This is reviewed in section 32.6 (see Derivation, Example 32.5, Example 32.6)

$$\mathcal{E} = -N\frac{d}{dt}\left(BA\cos(\omega t)\right) = NBA\omega\sin(\omega t)$$

  Here N is the number of turns of coil, B is the constant magnetic field, A is the area of the loop

In both cases you will need to do work to produce the current.

4.3 Inductance

- Consider a solenoid shown below in Fig. 4.1. The magnetic flux is proportional to the field, and the field is proportional to the current

$$\Phi_B \propto B \propto I \tag{4.6}$$

  The proportionality constant is known as the inductance or L

$$N\Phi_B \equiv LI \tag{4.7}$$

  Here N is the number of turns (see Fig. 4.1).

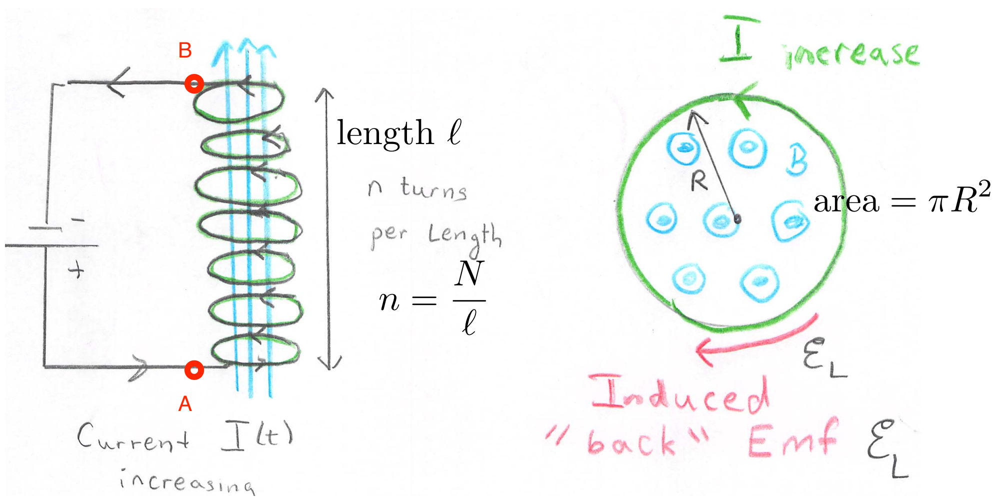


Figure 4.1: Inductance and back emf

- When the current is changed this produces a voltage called the "back emf"since it opposes the change in current. The voltage change across an inductor from point (A) to point (B) in Fig. 4.1 is

$$\Delta V = V_B - V_A = -L\frac{dI}{dt} \tag{4.8}$$

- You should know how to derive the inductance of a very long solenoid (you did it in class!)

$$L = \mu_o N^2\,\frac{\pi R^2}{\ell}$$

  where the number of turns per length is

$$n = \frac{N}{\ell} \tag{4.9}$$

  Start by recalling that the magnetic field in a solenoid is $B = \mu_o n I$.

- We showed that the (magnetic) energy stored in an inductor with current I is
$$\text{energy stored in the solenoid} = \mathcal{U}_B = \frac{1}{2}LI^2\,. \tag{4.10}$$
Using the field of a solenoid
$$B = \mu_o n I\,, \tag{4.11}$$
we find that the magnetic energy per volume is
$$u_B = \frac{\text{magnetic energy}}{\text{volume}} = \frac{\mathcal{U}_B}{\ell(\pi R^2)} = \frac{B^2}{2\mu_o}\,. \tag{4.12}$$
We see that this magnetic energy is entirely determined by the magnetic field and is independent of the geometry of the inductor.

- These formulas for the magnetic energy should be compared to the analogous formulas for the electric energy. For a capacitor of charge Q, plate area A, and plate separation d (see Fig. 4.2 below), the energy stored in a capacitor is
$$\mathcal{U}_E = \frac{Q^2}{2C} \tag{4.13}$$
where the capacitance is $C = \frac{1}{\epsilon_o}\frac{A}{d}$. Gauss Law can be used to relate the electric field to the charge stored
$$E = \frac{1}{\epsilon_o}\frac{Q}{A}\,, \tag{4.14}$$
Thus the electric energy per volume is
$$u_E = \frac{\text{electric energy}}{\text{volume}} = \frac{\mathcal{U}_E}{dA} = \frac{1}{2}\epsilon_o E^2 \tag{4.15}$$
We see that this electric energy is entirely determined by the electric field and is independent of the geometry of the capacitor.

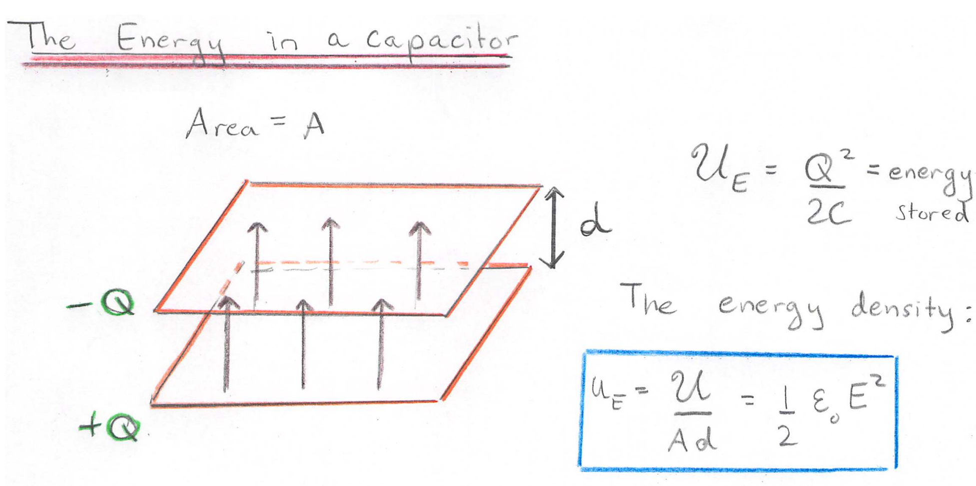


Figure 4.2: Capacitor energy

4.4 LR circuits – not on final but good to know

4.5 LC circuits

- A schematic of an LC circuit is shown below (see Fig. 4.3). A fully charged capacitor (with charge $Q_{\max}$) is connected to an inductor. When the switch is thrown, charge on the capacitor flows from the capacitor through the circuit. This builds up magnetic energy. Then the charge gathers on the opposite plate building up electric energy again. Then the charge runs through the system building up magnetic energy. Finally the charge returns to the original top plate. Then the process repeats itsself. The angular frequency of oscillation is

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{4.16}$$

So the period is

$$T = \frac{2\pi}{\omega_o} \tag{4.17}$$

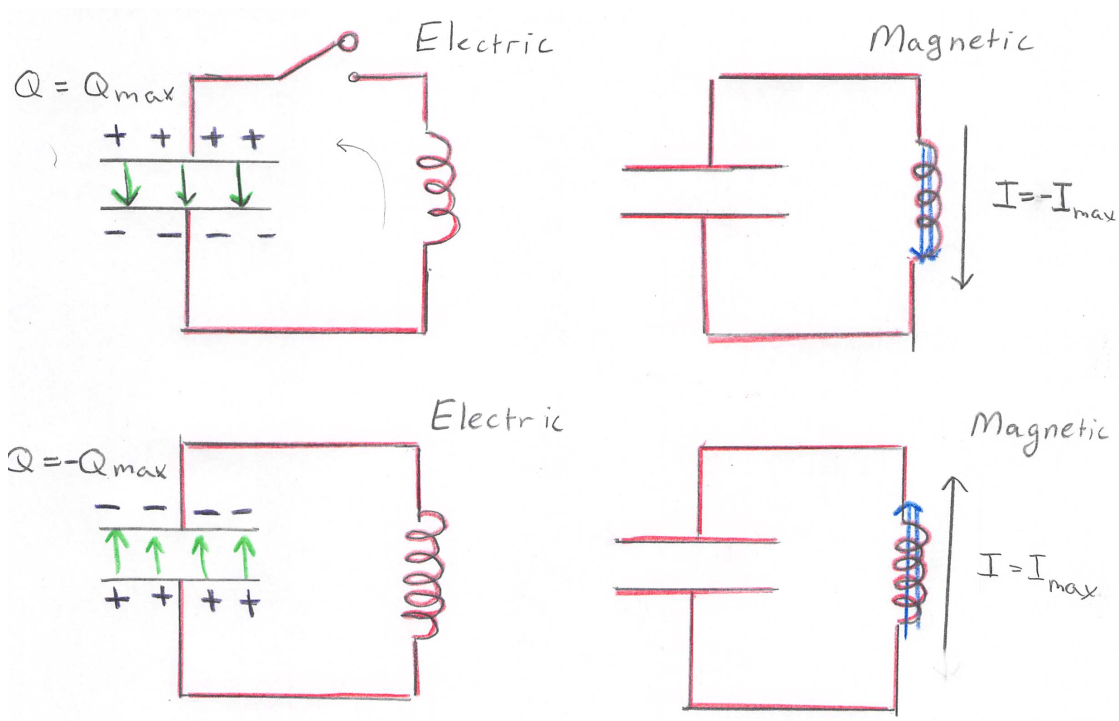


Figure 4.3: LC oscillations

- Applying Kirchoff Laws to the circuit we see (following the loop shown in the first panel of Fig. 4.3)

$$\Delta V_{\text{inductor}} + \Delta V_{\text{capacitor}} = 0 \tag{4.18a}$$

$$-L\frac{dI(t)}{dt} - \frac{Q(t)}{C} = 0 \tag{4.18b}$$

  where $Q(t)$ is the charge on the upper plate.

- The solution to Eq. (4.18b) is

$$Q(t) = Q_{\max}\cos(\omega_o t + \phi) \tag{4.19}$$

  where ϕ is the phase of the oscillation. If, as drawn in the Fig. 4.3, the charge is maximal at time $t = 0$ the phase is zero, $\phi = 0$. If the charge where maximal not at $t = 0$ but at another time, the phase would not be zero.

  You should feel comfortable determining and interpreting the current using $I = dQ(t)/dt$

$$I(t) = -I_{\max}\sin(\omega_o t) \qquad I_{\max} = \omega_o Q_{\max} \tag{4.20}$$

Be able to answer (in plain speak) the following questions: What is the voltage across the inductor versus time? What is the voltage across the capcitor versus time? What does a positive inductor voltage mean? What does a negative capacitor voltage mean?.

- The energy in an LC circuit oscillates between electric and magnetic energy. In class, you determined the electric energy and magnetic energy versus time. You should be able to graph these functions versus time.

  For instance, the electric energy is

$$u_E(t) = \frac{Q^2(t)}{2C} = \frac{Q^2_{\text{max}}}{2C}\cos^2(\omega_o t) \tag{4.21}$$

  Thus, the *maximum electric energy* is $Q^2_{\text{max}}/2C$, while the *average electric energy* is

$$\overline{u_E} = \frac{Q^2_{\text{max}}}{4C} \tag{4.22}$$

  You should be able to determine the magnetic energy versus time,

$$u_B = \frac{1}{2}LI^2(t) \tag{4.23}$$

  and be able to show that the maximum magnetic energy equals the maximum electric energy.

5 Maxwell correction and waves: Chapter 34

5.1 The maxwell correction to Amperes Law

- Maxwell reasoned that the Ampere's Law is incomplete the full statement is that the circulation of $\vec{\boldsymbol{B}}$ is determined by the current, I_{thru}, and the time derivative of the electric flux which acts like a current (known as the displacement current, I_D). In plain (but oversimplified) speak, changing electric fields make magnetic fields.

- The improved form is

$$\oint \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\ell}} = \mu_o(I_{\text{thru}} + I_D) \tag{5.1}$$

  where the *displacement* current is defined as the time derivative of the electric flux

$$I_D \equiv \epsilon_o \frac{d\Phi_E}{dt} \qquad \Phi_E \equiv \text{electric flux} \equiv \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}} \tag{5.2}$$

- You should understand the example shown in Fig. 5.1 that lead Maxwell to the notion of displacement current. In the left figure the through current I_{thru} is I_o. In the right figure there is no current passing

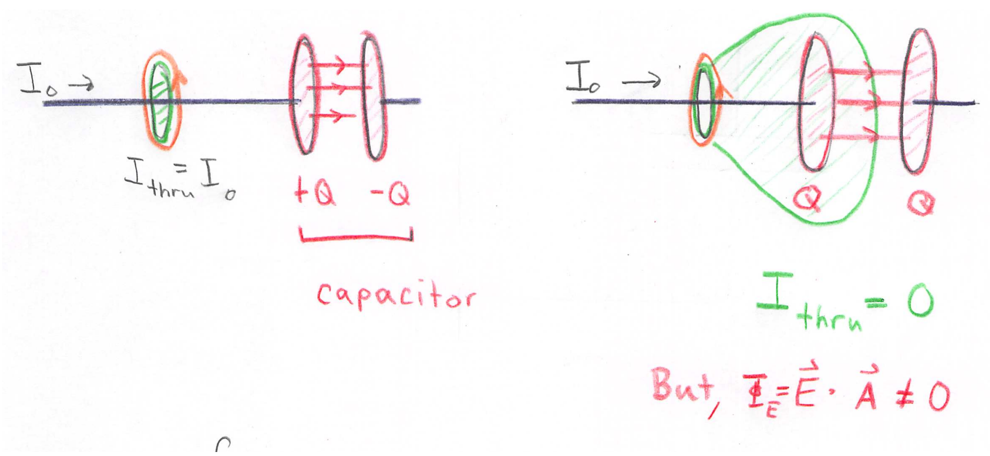


Figure 5.1: An example leading to the Maxwell correction

through the surface (since the charge builds up on the parallel plate capacitor and does not jump from one plate to the next), and $I_{\text{thru}} = 0$. But the circulation of $\vec{\boldsymbol{B}}$ around the (orange) loop (i.e. $\oint \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\ell}}$) is clearly the same in both cases. In the second case the electric flux through the surface is not zero and changing. The displacement current (the time derivative of the electric flux) is I_o. Thus the in both cases Eq. (5.1) is satisfied.

- Since changing magnetic fields make electric fields, and changing electric fields act like a current making magnetic fields, the two effects can be combined into a self supporting electro-magnetic waves.

Currents make magnetic fields; currents are moving charges; so changing magnetic fields are made by changing moving charges, i.e. accelerating charges. Thus propagating electromagnetic waves are started off by accelerating charges.

- The full Maxwell equations are

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_o} \qquad \text{(Gaus Law)} \tag{5.3}$$

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = 0 \qquad \text{(No isolated magnetic charges)} \tag{5.4}$$

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_o I + \mu_o \epsilon_o \frac{d\Phi_E}{dt} \qquad \text{(Ampere Law + Maxwell correction)} \tag{5.5}$$

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi_B}{dt} \qquad \text{Faraday Law} \tag{5.6}$$

together with the force law

$$\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{E}} + \boldsymbol{v} \times \vec{\boldsymbol{B}}) \tag{5.7}$$

We have studied each of these general laws in various forms.

- Maxwell showed that E and B in the absence of charges satisfy the wave equation, with wave speed

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{5.8}$$

and numerically found that $1/\sqrt{\mu_o \epsilon_o}$ is the speed of light.

5.2 Electromagnetic waves

- An electromagnetic wave consists of electric and magnetic fields which are perpendicular to each other and to the direction of propagation

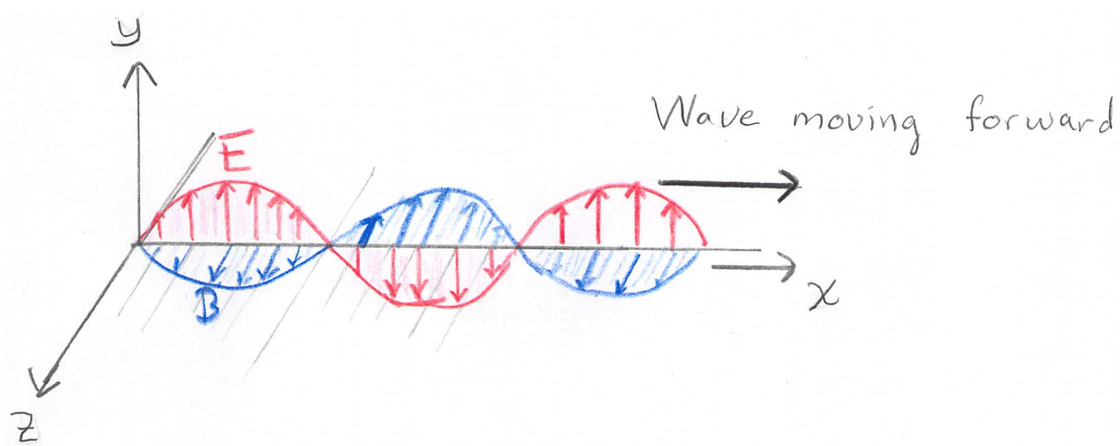


Figure 5.2: Electromagnetic waves

- For a wave propagating along the x direction with $\vec{\boldsymbol{E}}$ pointing in the y direction we have as shown in Fig. 5.2

$$\vec{\boldsymbol{E}}(t, x) = E_{\max} \sin(kx - \omega t)\, \hat{\mathbf{y}} \tag{5.9}$$

$$\vec{\boldsymbol{B}}(t, x) = B_{\max} \sin(kx - \omega t)\, \hat{\mathbf{z}} \tag{5.10}$$

- The wavelength, frequency and seed of light are related to the wavenumber, k, and the angular frequency, ω

$$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f \qquad c = \lambda f = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{5.11}$$

- The direction of propagation is given by the direction of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ as shown in Fig. 5.3. The magnitude of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ is discussed in the next section.

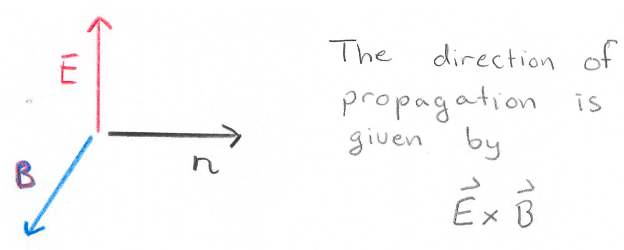


Figure 5.3: The direction of propagation and energy and momentum flow is given by the direction of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$.

- The amplitudes of the E and B fields are related

$$E_{\rm max} = c B_{\rm max}$$

5.3 Energy and momentum in electromagnetic waves

- The time averages of $\sin^2$ and $\cos^2$ are

$$\overline{\sin^2(\omega t + \phi)} = \overline{\cos^2(\omega t + \phi)} = \tfrac{1}{2}$$

since $\overline{\sin^2(x)} + \overline{\cos^2(x)} = 1$. For those who don't like the $\sin^2 + \cos^2 = 1$ "trick", you can do the integral and verify that

$$\overline{\sin^2(\omega t)} \equiv \frac{1}{T} \int_0^T \sin^2(\omega t) = \frac{1}{2} \tag{5.12}$$

where $T = 2\pi/\omega$ is a period of oscillation.

- Evaluating the electric energy density (energy per volume) we have that the time averaged electric energy density is

$$\overline{u_E} = \frac{1}{2} \epsilon_o \overline{E^2(t,x)} = \frac{1}{4} \epsilon_o E^2_{\rm max}\,, \tag{5.13}$$

we used that

$$\overline{E^2(t,x)} = E^2_{\rm max} \overline{\sin^2(kx - \omega t)} = \frac{1}{2} E^2_{\rm max} \tag{5.14}$$

while the magnetic energy density is

$$\overline{u_B} = \frac{1}{2\mu_o} \overline{B^2(t,x)} = \frac{1}{4\mu_o} B^2_{\rm max}\,. \tag{5.15}$$

- Show that the two energy densities are equal in magnitude $\overline{u_E} = \overline{u_B}$ since $cB_{\rm max} = E_{\rm max}$.Thus the total energy per volume is

$$\overline{u} = \overline{u_E} + \overline{u_B} = 2\overline{u_E} = 2\overline{u_B} = \frac{1}{2} \epsilon_o E^2_{\rm max} = \frac{1}{2\mu_o} B^2_{\rm max} \tag{5.16}$$

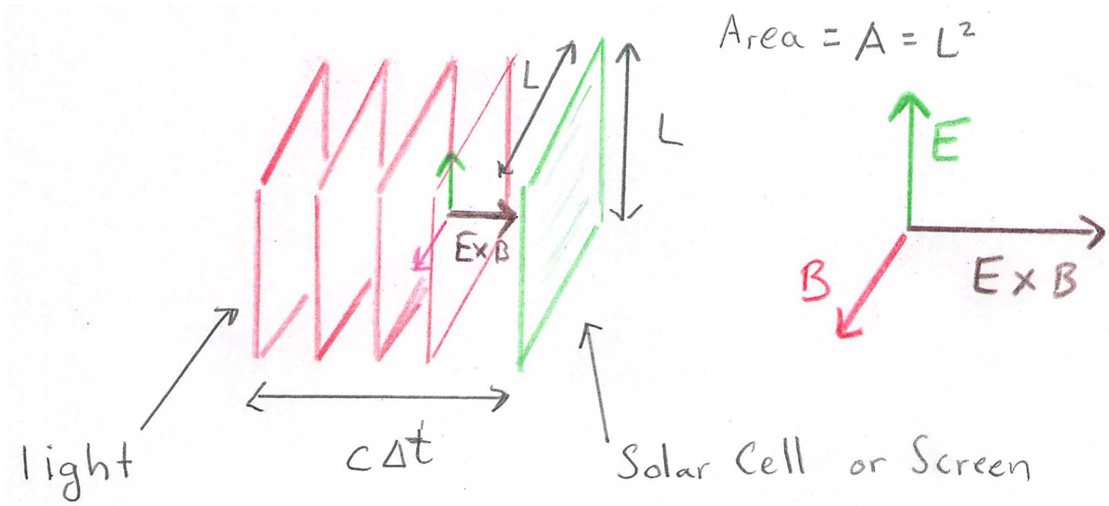


Figure 5.4: Light of average energy density $\overline{u}$ and volume $Ac\Delta t$ flowing through the screen

- The intensity, $I \equiv \overline{S}$, is the energy per area per time delivered by the light. You should understand Fig. 5.4 and be able to show that the energy crossing the screen per area per time is related to the energy per volume

$$I \equiv \overline{S} = \frac{1}{A}\frac{\Delta E}{\Delta t} = \overline{u}\, c \tag{5.17}$$

- We introduced the Poynting vector

$$\vec{\boldsymbol{S}} = \frac{\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}}{\mu_o} \tag{5.18}$$

  which points in the direction of energy flow.

  (a) The magnitude of the Poynting vector is the instantaneous flow of energy per area per time, i.e. the energy flowing across a screen (of area $\vec{\boldsymbol{A}}$) per unit time is

$$\frac{\Delta U}{\Delta t} = \vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{A}} = SA\cos\theta \tag{5.19}$$

  Thus, because of the $\cos\theta$ in the dot product, if sunlight strikes a surface head on more energy is absorbed per time. In the summer sunlight strikes the earth head on, while in the winter sunlight is absorbed only at an angle since the sun is low in the sky due to the tilt of the earth.

  (b) We showed that time averaged magnitude of the Poynting vector is the intensity

$$\overline{S} = \overline{u}\, c \tag{5.20}$$

- The average momentum per volume in a light wave is

$$\overline{g} = \frac{\overline{u}}{c} = \frac{S}{c^2} \tag{5.21}$$

  and more generally the momentum per volume is $\vec{\boldsymbol{g}} = \vec{\boldsymbol{S}}/c^2$.

  (a) Thus, if a screen (or particle, or solar cell,...) absorbs a certain energy $\Delta U = \overline{u}\Delta V$ from a light wave, then momentum it receives is

$$\Delta P = \frac{\Delta U}{c} \tag{5.22}$$

  where $\Delta P = \overline{g}\Delta V$.

(b) We also showed using the Fig. 5.4 that the momentum absorbed per area per time ($1/A\Delta P/\Delta t$) is $\overline{g}\,c$. You should be able to derive this from Fig. 5.4.

Since momentum absorbed per area per time is the force per area, the radiation pressure (i.e. the pressure felt by a screen such as Fig. 5.4 absorbing light) is

$$\frac{F}{A} = \frac{1}{A}\frac{\Delta P}{\Delta t} = \overline{g}c \qquad \text{absorbing light} \tag{5.23}$$

Sometimes the light is not absorbed but reflected, in this case the radiation pressure is twice the previous case

$$\frac{F}{A} = 2\overline{g}c \qquad \text{reflecting light} \tag{5.24}$$

since the light comes in and bounces back.