In this note we will find the green function of one-dimensional second order differential equations

17 and 18

Second Order DEQ and Green fcns

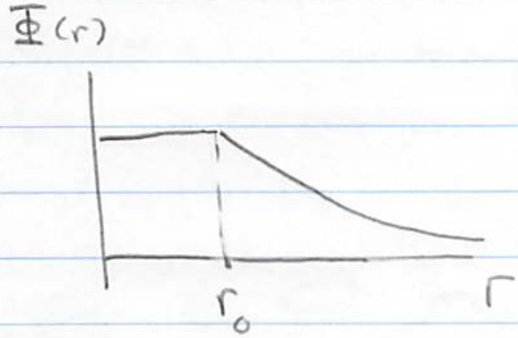
- The procedure to constructi the green function is always the same as in the previous example
    - Solve the homogeneous equations to the left and right
    - integrate across the δ-fcn to match the two sols

Let us follow this procedure for the equation

$$\left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] G(x,x_0) = \delta(x - x_0)$$

- For definiteness take a specific problem. The potential of a charged spherical shell

$$\Phi(r, r_0) = \begin{cases} \dfrac{Q}{4\pi r_0} & r < r_0 \\ \dfrac{Q}{4\pi r} & r > r_0 \end{cases}$$



This is the solution to

$$\nabla \cdot E = \rho \qquad E = -\nabla \Phi$$

$$-\nabla^2 \Phi = \rho$$

Where $\rho(r) = \dfrac{Q}{4\pi r^2}\,\delta(r - r_0)$ ← spherical shell of radius r_0 and charge Q

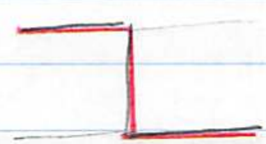
$$-\nabla^2 \Phi = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} = \frac{Q}{4\pi}\frac{1}{r^2}\delta(r - r_0)$$

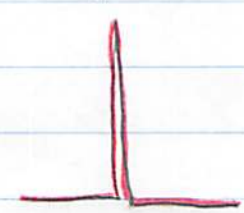
So we see that we are trying to find the Green function of

$$-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} G(r, r_0) = \delta(r - r_0) \qquad \Phi = \frac{Q}{4\pi} G(r, r_0)$$

- Note we are looking for a solution that looks like this:

$y(x) =$  ← kink

$y'(x) =$  ← θ fcn

$y''(x) =$  ← δ-fcn

So the function is continuous, but has a discontinuous derivative. We can see this directly from the general EOM

Integrating from $x_0 - \varepsilon$ to $x_0 + \varepsilon$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \left[-\frac{d}{dx}\left(p(x) \frac{dG}{dx}\right) + q(x)\, G \right] = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)$$

Yielding

electric field in our example

$$- p(x) \frac{dG}{dx}\Bigg|_{x=x_0+\varepsilon} + p(x) \frac{dG}{dx}\Bigg|_{x=x_0-\varepsilon} = 1 \qquad (20.1)$$

Jump condition

For the simple case of a charged sphere this says that the jump in the electric field is due to the surface charge.

$$-r^2 \frac{\partial G}{\partial r}\Bigg|_{r=r_0+\varepsilon} - \left(-r^2 \frac{\partial G}{\partial r}\right)\Bigg|_{r=r_0-\varepsilon} = 1 \qquad (20.2)$$

or since $G = \Phi / (Q/4\pi)$ and $E_r = -\partial\Phi/\partial r$

$$4\pi r^2 E_r \Big|_{out} - 4\pi r^2 E_r \Big|_{in} = Q \qquad (20.3)$$

- Now we have analyzed the "jump condition" which relates the interior and exterior solutions. We now should solve for these solutions

- Let the homogeneous solution to the left $(x < x_0)$ be $y_<(x)$ and the solution to the right be $y_>(x)$ (i.e $x > x_0$). For our case we solve

$$-\frac{d}{\partial r} r^2 \frac{\partial}{\partial r} \Phi = 0 \qquad \begin{cases} \text{b.c. regular as } r \to 0 \\ \text{and vanishes as } r \to \infty \end{cases}$$

There are two solutions 1 and $1/r$

$$y(r) = C_1 \cdot 1 + C_2 \cdot \frac{1}{r}$$

- Our solutions obey homogeneous b.c. (why?) So one of these constants can be chosen at will since if y satisfies the DEQ and b.c., then so does $c\,y(r)$. Here then, $y_<(r) = 1$ and $y_>(r) = \frac{1}{r}$

These follow from our b.c., that $G(r, r_0)$ be regular ar $r \to 0$ and $r \to \infty$.

- The Green function then takes the form

$$G(x, x_0) = C_1\, y_<(x)\, \Theta(x_0 - x) + C_2\, y_>(x)\, \Theta(x - x_0)$$

Continuity Givers at $r = r_0$ the condition:

(21.1) $$G(x, x_0) = C\left[y_<(x)\, y_>(x_0)\, \Theta(x_0 - x) + y_>(x)\, y_<(x_0)\, \Theta(x - x_0) \right]$$

- Now we can determine the remaining coefficient C from the jump condition Eq (20,1)

Substituting Eq (21.1) into Eq (20.1) gives (Do it!!!)

$$- p(x)\, C\, y_>'(x)\, y_<(x_0) \Big|_{x=x_0+\varepsilon} + p(x)\, C\, y_<'(x)\, y_>(x_0) \Big|_{x=x_0-\varepsilon} = 1$$

Or

$$C = \frac{1}{p(x_0)\,[\,y_>(x_0)\, y_<'(x_0) - y_>'(x_0)\, y_<(x_0)\,]}$$

i.e.

$$C = \frac{1}{p(x_0)\, W(x_0)} \qquad \underbrace{W(x_0) = y_> y_<' - y_< y_>'}_{\text{Wronskian of } y_>, y_<}$$

recall that this is constant (Eq 6.1)

and

$$\boxed{G(x, x_0) = \frac{y_>(x)\, y_<(x_0)\, \Theta(x-x_0) + y_<(x)\, y_>(x_0)\, \Theta(x_0-x)}{p(x_0)\, W(x_0)}}$$

Eq (22.1)

the green fcn for a general 2nd order DEQ.

the denom is a constant indep of x_0

- For our particular example

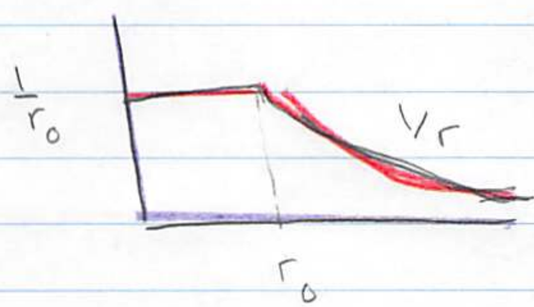
$$y_> = 1/r \qquad y_< = 1 \qquad p(r) = r^2$$

$$W(r) = \cancel{y_> y'_<} - y_< y'_> = -1 \cdot \frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}$$

And $p(r)\, W(r) = 1$ thus

$$G(r, r_0) = \frac{1}{r} \cdot 1 \; \Theta(r - r_0) + 1 \cdot \frac{1}{r_0} \; \Theta(r_0 - r) \quad \checkmark$$

- Graph



as expected!
This is the potential of a charged sphere up to $Q/4\pi$