1 Introduction to the Maxwell Equations

1.1 The maxwell equations and units

- We use Heavyside Lorentz system of units. This is discussed in a separate note.
- The Maxwell equations are

$$\nabla \cdot \boldsymbol{E} = \rho \tag{1.1}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{1.2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.3}$$

$$-\nabla \times \boldsymbol{E} = \frac{1}{c}\partial_t \boldsymbol{B} \tag{1.4}$$

In integral form we have

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = Q_{\rm enc} \tag{1.5}$$

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{I}{c} + \frac{1}{c}\partial_t \Phi_E \tag{1.6}$$

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0 \tag{1.7}$$

$$-\oint_\ell \boldsymbol{E} \cdot d\boldsymbol{\ell} = \frac{1}{c}\partial_t \Phi_B \tag{1.8}$$

Here $\Phi_E = \int \boldsymbol{E} \cdot d\boldsymbol{S}$ is the electric flux, $\Phi_B = \int \boldsymbol{B} \cdot d\boldsymbol{S}$ is the magnetic flux, and $I = \int_S \boldsymbol{j} \cdot d\boldsymbol{S}$ is the current crossing a surface, S. $d\boldsymbol{S}$ is the surface element with a specified area and normal $d\boldsymbol{S} = \boldsymbol{n}\, d(\text{area})$. $d\boldsymbol{\ell}$ denotes a closed line integral element.

- We specify the currents and solve for the fields. In media we specify a constituent relation relating the current to the electric and magnetic fields.
- The Maxwell force law

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \tag{1.9}$$

- Hemholtz Theorems state:

(a) Given a curl free vecor field, $\boldsymbol{C}(\boldsymbol{x})$, there exsists a scalar function, $S(\boldsymbol{x})$, such that $\boldsymbol{C} = -\nabla S$:

$$\text{if} \quad \nabla \times \boldsymbol{C}(\boldsymbol{x}) = 0 \quad \text{then} \quad \boldsymbol{C} = -\nabla S(\boldsymbol{x}) \tag{1.10}$$

(b) Given a divergence free vector field, $\boldsymbol{D}(\boldsymbol{x})$, there exsists a vector field $\boldsymbol{V}$ such that $\boldsymbol{D} = \nabla \times \boldsymbol{V}$:

$$\text{if} \quad \nabla \cdot \boldsymbol{D}(\boldsymbol{x}) = 0 \quad \text{then} \quad \boldsymbol{D} = \nabla \times \boldsymbol{V}(\boldsymbol{x}) \tag{1.11}$$

The converses are easily proved, $\nabla \times \nabla S(\boldsymbol{x}) = 0$, and $\nabla \cdot \nabla \times \boldsymbol{V}(\boldsymbol{x}) = 0$ There are two *very important* consequences for the Maxwell equations.

(a) From the source free Maxwell equations (eqs. three and four) one finds that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.12}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\phi \tag{1.13}$$

(b) Current conservation follows by manipulating the sourced maxwell equations (eqs. one and two)

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0 \tag{1.14}$$

- For a system of characteristic length L (say one meter) and characteristic time scale T (say one second), we can expand the fields in $1/c$ since $(L/T)/c \ll 1$:

$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \ldots \tag{1.15}$$

$$\boldsymbol{B} = \boldsymbol{B}^{(0)} + \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \ldots \tag{1.16}$$

where each term is smaller than the next by $(L/T)/c$. At zeroth order we have

$$\nabla \cdot \boldsymbol{E}^{(0)} = \rho \tag{1.17}$$

$$\nabla \times \boldsymbol{E}^{(0)} = 0 \tag{1.18}$$

$$\nabla \cdot \boldsymbol{B}^{(0)} = 0 \tag{1.19}$$

$$\nabla \times \boldsymbol{B}^{(0)} = 0 \tag{1.20}$$

These are the equations of electro statics. Note that $\boldsymbol{B}^{(0)} = 0$ to this order (for a field which is zero at infinity)

- At first order we have

$$\nabla \cdot \boldsymbol{E}^{(1)} = 0 \tag{1.21}$$

$$\nabla \times \boldsymbol{E}^{(1)} = 0 \qquad (\text{ since } \partial_t \boldsymbol{B}^{(0)} = 0 \text{ }) \tag{1.22}$$

$$\nabla \cdot \boldsymbol{B}^{(1)} = 0 \tag{1.23}$$

$$\nabla \times \boldsymbol{B}^{(1)} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E}^{(0)} \tag{1.24}$$

This is the equation of magneto statics, with the contribution of the Maxwell term, $1/c\,\partial_t \boldsymbol{E}^{(0)}$, computed with electrostatics. Note that $\boldsymbol{E}^{(1)} = 0$

2 Electrostatics

2.1 Elementary Electrostatics

Electrostatics:

(a) Fundamental Equations

$$\nabla \cdot \boldsymbol{E} = \rho \tag{2.1}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{2.2}$$
$$\boldsymbol{F} = q\boldsymbol{E} \tag{2.3}$$

(b) Given the divergence theorem, we may integrate over volume of $\nabla \cdot \boldsymbol{E} = \rho$ and deduce Gauss Law:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = q_{\rm tot}$$

which relates the flux of electric field to the enclosed charge

(c) For a point charge $\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ and the field of a point charge

$$\boldsymbol{E} = \frac{q\,\widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{2.4}$$

and satisfies

$$\nabla \cdot \frac{q\,\widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) \tag{2.5}$$

(d) The potential. Since the electric field is curl free (in a quasi-static approximation) we may write it as gradient of a scalar

$$\boldsymbol{E} = -\nabla\Phi \qquad \Phi(\boldsymbol{x}_b) - \Phi(\boldsymbol{x}_a) = -\int_a^b \boldsymbol{E} \cdot {\rm d}\boldsymbol{\ell} \tag{2.6}$$

The potential satisfies the Poisson equation

$$-\nabla^2\Phi = \rho\,. \tag{2.7}$$

The Laplace equation is just the homogeneous form of the Poisson equation

$$-\nabla^2\Phi = 0. \tag{2.8}$$

The next section is devoted to solving the Laplace and Poisson equations

(e) The boundary conditions of electrostatics

$$\boldsymbol{n} \cdot (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \sigma \tag{2.9}$$
$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \tag{2.10}$$

i.e. the components perpendicular to the surface (along the normal) jump, while the parallel components are continuous.

(f) The Potential Energy stored in an ensemble of charges is

$$U_E = \frac{1}{2}\int \mathrm{d}^3x\, \rho(\boldsymbol{r})\Phi(\boldsymbol{r}) \tag{2.11}$$

(g) The energy density of an electrostatic field is

$$u_E = \frac{1}{2}E^2 \tag{2.12}$$

(h) Force and stress

i) The stress tensor records T^{ij} records the force per area. It is the force in the j-th direction per area in the i-th. More precisely let $\boldsymbol{n}$ be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, } \textit{by region LEFT on region RIGHT} \tag{2.13}$$

ii) The total momentum density $\mathbf{g}_{tot}$ (momentum per volume) is supposed to obey a conservation law

$$\partial_t g^j_{\rm tot} + \partial_i T^{ij} = 0 \qquad \partial_t g^j_{\rm tot} = -\partial_i T^{ij} \tag{2.14}$$

Thus we interpret the net force per volume f^j as the (negative) divergence of the stress

$$f^j = -\partial_i T^{ij} \tag{2.15}$$

iii) The stress tensor of a gas or fluid at rest is $T^{ij} = p\delta^{ij}$ where p is the pressure, so the force per volume $\boldsymbol{f}$ is the negative gradient of pressure.

iv) The stress tensor of an electrostatic field is

$$T^{ij}_E = -E^iE^j + \tfrac{1}{2}\delta^{ij}E^2 \tag{2.16}$$

Note that I will use an opposite sign convention from Jackson: $T^{ij}_{\rm me} = -T^{ij}_{\rm Jackson}$. This convention has some good features when discussing relativity.

v) The net electric force on a charged object is

$$F^j = \int \mathrm{d}^3x\, \rho(\boldsymbol{r})E^j(\boldsymbol{r}) = -\int dS\, n_i T^{ij} \tag{2.17}$$

(i) For a metal we have the following properties

i) On the surface of the metal the electric field is normal to the surface of the metal. The charge per area σ is related to the magnitude of the electric field. Let $\boldsymbol{n}$ be pointing from inside to outside the metal:

$$\boldsymbol{E} = E_n\boldsymbol{n} \qquad \sigma = E_n \tag{2.18}$$

ii) Forces on conductors. In a conductor the force per area is

$$\mathcal{F}^i = \frac{1}{2}\sigma E^i = \frac{1}{2}\sigma_n^2\, n^i \tag{2.19}$$

The one half arises because half of the surface electric field arises from σ itself, and we should not include the self-force. This can also be computed using the stress tensor

iii) Capacitance and the capacitance matrix and energy of system of conductors
For a single metal surface, the charge induced on the surface is proportional to the Φ.

$$q = C\Phi\,.$$

When more than one conductor is involved this is replaced by the matrix equation:

$$q_A = \sum_B C_{AB}\Phi_B\,.$$

2.2 Multipole Expansion

Cartesian and Spherical Multipole Expansion

(a) **Cartesian Multipole expansion**

For a set of charges in 3D arranged with characteristic size L, the potential far from the charges $r \gg L$ is expanded in *cartesian multipole* moments

$$\Phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(r_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{2.20}$$

$$\Phi(\boldsymbol{r}) \simeq \frac{1}{4\pi}\left[\frac{q_{tot}}{r} + \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} + \tfrac{1}{2}\mathcal{Q}_{ij}\frac{\hat{\boldsymbol{r}}^i\hat{\boldsymbol{r}}^j}{r^3} + \ldots\right] \tag{2.21}$$

where each terms is smaller than the next since r is large. Here monopole moment, the dipole moment, and (traceless) quadrupole moments are respectively:

$$q_{\rm tot} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r}) \tag{2.22}$$

$$\boldsymbol{p} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r})\boldsymbol{r} \tag{2.23}$$

$$\mathcal{Q}_{ij} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r})\left(3r_ir_j - \boldsymbol{r}^2\delta_{ij}\right) \tag{2.24}$$

respectively. There are five independent components of the symmetric and traceless tensor (matrix) $\mathcal{Q}_{ij}$. We have implicitly defined the moments with respect to an agreed upon origin $\boldsymbol{r}_o = \boldsymbol{0}$.

(b) **Forces and energy of a small charge distribution in an external field**

Given an external field $\Phi(\boldsymbol{r})$ we want to determine the energy of a charge distribution $\rho(\boldsymbol{r})$ in this external field. The potential energy of the charge distribution is

$$U_E = Q_{\rm tot}\Phi(\boldsymbol{r}_o) - \boldsymbol{p}\cdot\boldsymbol{E}(\boldsymbol{r}_o) - \frac{1}{6}\mathcal{Q}^{ij}\partial_iE_j(\boldsymbol{r}_o) + \ldots \tag{2.25}$$

where $\boldsymbol{r}_o$ is a chosen point in the charge distribution and the $Q_{\rm tot}, \boldsymbol{p}, \mathcal{Q}^{ij}$ are the multipole moments around that point (see below).

The multipoles are defined around the point $\boldsymbol{r}_o$ on the small body:

$$Q_{\rm tot} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r}) \tag{2.26}$$

$$\boldsymbol{p} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r})\, \delta\boldsymbol{r} \tag{2.27}$$

$$\mathcal{Q}_{ij} = \int \mathrm{d}^3x\, \rho(\boldsymbol{r})\left(3\,\delta r_i\,\delta r_j - \delta\boldsymbol{r}^2\,\delta_{ij}\right) \tag{2.28}$$

where $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_o$.

The force on a charged object can be found by differentiating the energy

$$\boldsymbol{F} = -\nabla_{\boldsymbol{r}_o}U_E(\boldsymbol{r}_o) \tag{2.29}$$

For a dipole this reads

$$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E} \tag{2.30}$$

(c) **Spherical multipoles.** To determine the potential far from the charge we we determine the potential to be

$$\Phi(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(r_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{2.31}$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{q_{\ell m}}{2\ell+1}\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \tag{2.32}$$

Now we characterize the charge distribution by spherical multipole moments:

$$q_{\ell m} = \int d^3\boldsymbol{r}_o\, \rho(\boldsymbol{r}_o)\ \left[r_o^\ell\, Y^*_{\ell m}(\theta_o, \phi_o)\right] \tag{2.33}$$

You should feel comfortable deriving this using an identity we derived in class (and will further discuss later)

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} = \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta_o,\phi_o) \tag{2.34}$$

Here

$$r_> = \text{greater of } r \text{ and } r_o \tag{2.35}$$

$$r_< = \text{lesser of } r \text{ and } r_o \tag{2.36}$$

$$\tag{2.37}$$

Could also notate this as

$$\frac{r_<^\ell}{r_>^{\ell+1}} = \frac{r_o^\ell}{r^{\ell+1}}\theta(r-r_o) + \frac{r^\ell}{r_o^{\ell+1}}\theta(r_o - r)\,. \tag{2.38}$$

I find this form clearer, since I know how to differntiate the right hand side using, $d\theta(x-x_o)/dx = \delta(x-x_o)$

(d) For an azimuthally symmetric distribution only $q_{\ell 0}$ are non-zero, the equations can be simplified using $Y_{\ell 0} = \sqrt{(2\ell+1)/4\pi}P_\ell(\cos\theta)$ to

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \tag{2.39}$$

(e) There is a one to one relation between the cartesian and spherical forms

$$p_x, p_y, p_z \leftrightarrow q_{11}, q_{10}, q_{1-1} \tag{2.40}$$

$$\mathcal{Q}_{zz}, \mathcal{Q}_{xx} - \mathcal{Q}_{yy}, \mathcal{Q}_{xy}, \mathcal{Q}_{zx}, \mathcal{Q}_{zy} \leftrightarrow q_{22}, q_{21}, q_{20}, q_{2-1}, q_{2-2} \tag{2.41}$$

which can be found by equating Eq. (2.31) and Eq. (2.20) using

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.42}$$

3 Mathematics of the Poisson Equation

3.1 Green functions and the Poisson equation

(a) The Dirichlet Green function satisfies the Poisson equation with delta-function charge

$$-\nabla^2 G_D(\boldsymbol{r}, \boldsymbol{r}_o) = \delta^3(\boldsymbol{r} - \boldsymbol{r}_o) \tag{3.1}$$

and vanishes on the boundary. *It is the potential at $\boldsymbol{r}$ due to a point charge (with unit charge) at $\boldsymbol{r}_o$ in the presence of grounded ($\Phi = 0$) boundaries* The simplest free space green function is just the point charge solution

$$G_o = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{3.2}$$

In two dimensions the Green function is

$$G_o = \frac{-1}{2\pi} \log|\boldsymbol{r} - \boldsymbol{r}_o| \tag{3.3}$$

which is the potential from a line of charge with charge density $\lambda = 1$

(b) With Dirichlet boundary conditions the Laplacian operator is self-adjoint. The dirichlet Green function is symmetric $G_D(\boldsymbol{r}, \boldsymbol{r}_0) = G_D(\boldsymbol{r}_0, \boldsymbol{r})$. This is known as the Green Reciprocity Theorem, and appears in many clever ways.

The intuitive way to understand this is that for grounded boundary b.c. $-\nabla^2$ is a real self adjoint operator (i.e. a real symmetric matrix). Now $-\nabla^2 G_D(\boldsymbol{r}, \boldsymbol{r}_0) = \delta^3(\boldsymbol{r} - \boldsymbol{r}_0)$, so in a functional sense $G_D(\boldsymbol{r}, \boldsymbol{r}_0)$ is the inverse matrix of $-\nabla^2$. The inverse of a real symmetric matrix is also real and symmetric. If the Laplace equation with Dirichlet b.c. is discretized for numerical work, these statements become explicitly rigorous.

(c) The Poisson equation or the boundary value problem of the Laplace equation can be solved once the Dirichlet Green function is known.

$$\Phi(\boldsymbol{r}) = \int_V \mathrm{d}^3x_o \, G_D(\boldsymbol{r}, \boldsymbol{r}_o)\rho(\boldsymbol{r}_o) - \int_{\partial V} dS_o \, \boldsymbol{n}_o \cdot \nabla_{r_o} G_D(\boldsymbol{r}, \boldsymbol{r}_o)\Phi(r_o) \tag{3.4}$$

where $\boldsymbol{n}_o$ is the *outward* directed normal. The first term is a volume integral and is the contribution of the interior charges on the potential. The second term is a surface integral, and is the contribution of the boundary value to the interior.

(d) A useful technique to find a Green function is image charges. You should know the image charge green functions

i) A plane in 1D and 2D (class)

ii) A sphere (homework)

iii) A cylinder (homework + recitation)

(e) The Green function can always be written in the form

$$G(\boldsymbol{r},\boldsymbol{r}_o) = \underbrace{G_o(\boldsymbol{r},\boldsymbol{r}_o)}_{\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}} + \Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_o) \tag{3.5}$$

where the induced potential, $\Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_o)$, is regular and satisfies the homogeneous equation $-\nabla^2\Phi_{\rm ind} = 0$.

The force of a point charge q and the grounded boundaries (*i.e.* between the charge q and the induced charges on the grounded surfaces) is entirely due to the induced potential[1]

$$\boldsymbol{F} = -\,q^2 \nabla_{\boldsymbol{r}} \Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_0)\Big|_{\boldsymbol{r}=r_0} \tag{3.6}$$

Using the green reciprocity theoem $\Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_0) = \Phi_{\rm ind}(\boldsymbol{r}_0,\boldsymbol{r})$, we can write[2]

$$\boldsymbol{F} = -\nabla_{\boldsymbol{r}_0} U_{\rm int}(\boldsymbol{r}_0) \tag{3.8}$$

where

$$U_{\rm int}(r_o) = \frac{1}{2}q^2\Phi_{\rm ind}(\boldsymbol{r}_o,\boldsymbol{r}_o) = \frac{1}{2}q^2 \lim_{\boldsymbol{r}\to\boldsymbol{r}_o} \left(G(\boldsymbol{r},\boldsymbol{r}_o) - G_{\rm o}(\boldsymbol{r},\boldsymbol{r}_o)\right) \tag{3.9}$$

(f) Finding the Green function by separation of variables This is best illustrated by example. Pick two dimensions of a surface (say θ,ϕ). The method is motivated by the fact that $\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)$ can be written as a sum

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = \frac{1}{r^2}\delta(r-r_o)\delta(\cos\theta-\cos\theta_o)\delta(\phi-\phi_o) = \frac{1}{r^2}\delta(r-r_o)\sum_{\ell m} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o,\phi_o) \tag{3.10}$$

Thus the green function is can also be written as

$$G(\boldsymbol{r},\boldsymbol{r}_o) = \sum_{\ell=0}^{\infty}\sum_{-\ell}^{\ell} g_{\ell m}(r,r_o) Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o,\phi_o) \tag{3.11}$$

leading to an equation for $g_{\ell m}(r,r_o)$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}\right] g_{\ell m}(r,r_o) = \frac{1}{r^2}\delta(r-r_o) \tag{3.12}$$

This remaining equation in 1D is then solved for the green function following the strategy outlined in Sect. 3.2 (see Eq. (3.37)). This depends on the conditions boundary conditions. Similar expressions can be derived in other coordinates.

(g) For free space, the two solutions to Eq. (3.12) are $y_{out}(r) = 1/r^{\ell+1}$ and $y_{\rm in}(r) = r^\ell$, $p(r) = r^2$ and $p(r)W(r) = 2\ell+1$. Then the free space Green fcn can be written

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} = \sum_{\ell=0}^{\infty}\sum_{-\ell}^{\ell}\left[Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o,\phi_o)\right]\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} \tag{3.13}$$

Some useful identities can be derived from Eq. (3.13):

[1] The green function is the potential for a unit charge $q=1$. The induced charges are proportional to q. The electro-static field from the induced charges is $\boldsymbol{E}_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_0) = -q\nabla_{\boldsymbol{r}}\Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_o)$ while the force on q is $\boldsymbol{F} = q\boldsymbol{E}_{\rm ind}(\boldsymbol{r}_0,\boldsymbol{r}_0)$.

[2] We use

$$\nabla_{\boldsymbol{r}}\Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_0)\Big|_{\boldsymbol{r}=\boldsymbol{r}_0} = \frac{1}{2}\Big[\nabla_{\boldsymbol{r}}\Phi_{\rm ind}(\boldsymbol{r},\boldsymbol{r}_0) + \nabla_{\boldsymbol{r}}\Phi_{\rm ind}(\boldsymbol{r}_0,\boldsymbol{r})\Big]_{\boldsymbol{r}=\boldsymbol{r}_0} = \frac{1}{2}\nabla_{\boldsymbol{r}_0}\Phi_{\rm ind}(\boldsymbol{r}_0,\boldsymbol{r}_0) \tag{3.7}$$

i) The generating function of Legendre Polynomials is found by setting $\boldsymbol{r}_o = \hat{\boldsymbol{z}}$ and $r < 1$ with $Y_{\ell 0} = \sqrt{(2\ell+1)/4\pi} P_\ell(\cos\theta)$

$$\frac{1}{\sqrt{1+r^2-2r\cos\theta}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta) \tag{3.14}$$

ii) The spherical harmonic addition theorem which we find by writing by setting $\boldsymbol{r}_o = 1$ and $r < 1$ and using $1/|\boldsymbol{r} - \boldsymbol{r}_o| = 1/\sqrt{1 + r^2 - 2r\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o}$

$$P_\ell(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o\phi_o) \tag{3.15}$$

where $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o$ is the cosine of the angle between the two vectors.

iii) The shell structure relation which you find by setting $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}_o$

$$1 = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta,\phi) \tag{3.16}$$

This relation is what is responsible for shell structure in the periodic table

(h) Similar expansion exists in other coordinates. *e.g.* in cylindrical coords $y_{out}(\rho) = K_m(\kappa\rho)$ and $y_{in}(\rho) = I_m(\kappa\rho)$, leading to

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int\frac{dk}{2\pi}\left[e^{im(\phi-\phi_o)}e^{ik(z-z_o)}\right] I_m(k\rho_<)K_m(k\rho_>) \tag{3.17}$$

3.2 Solving the Laplace Equation by Separation

A summary of separation of variables in different coordinate systems is given in Appendix D. The most important case is spherical and cartesian coordinates.

Solving the Laplace equation

We use a technique of separation of variables in different coordinate systems. The technique of separation of variables is best illustrated by example. For instance consider a potential in a square geometry. The

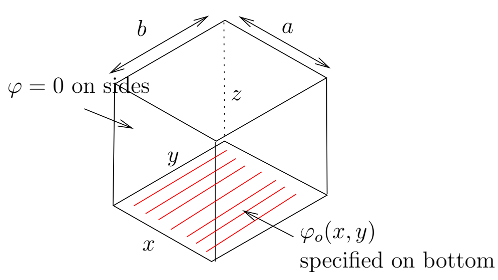


Figure 3.1: A rectangle illustrating separation of vars

potential $\Phi(x, y, z)$ is specified at $z = 0$ to be $\Phi_o(x, y)$ and zero on the remaining boundaries

(a) We look for solutions of the separated form

$$\Phi = \underbrace{Z(z)}_{\perp\text{ to surf}}\ \underbrace{X(x)Y(y)}_{\parallel\text{ to surf}} \tag{3.18}$$

Substituting this into the laplace equation, and separating variables gives two equations for X, Y (the parallel directions)

$$\left[-\frac{d^2}{dx^2} - k_n^2\right] X(x) = 0\,, \tag{3.19}$$

$$\left[-\frac{d^2}{dy^2} - k_m^2\right] Y(x) = 0\,. \tag{3.20}$$

and one equation for the perpendicular equation

$$\left[-\frac{d^2}{dz^2} + k_z^2\right] Z(z) = 0\,, \tag{3.21}$$

where $k_z^2 = k_n^2 + k_m^2$. The signs of k_x, k_y, k_z are chosen for later convenience, because it will be impossible to satisfy the BC for $k_x^2 < 0$ or $k_y^2 < 0$.

The first step is always to separate variables and write down the general solutions to the separated equations

$$X(x) = A\cos(k_n x) + B\sin(k_n x) \tag{3.22}$$

$$Y(y) = A\cos(k_m y) + B\sin(k_m y) \tag{3.23}$$

$$Z(z) = Ae^{-k_z z} + Be^{k_z z} \tag{3.24}$$

(b) It is best to analyze the parallel equations first which are all of the form of a Sturm Louiville *eigenvalue* equation (see below). These determine the (eigen) functions $X(x), Y(y)$ *and* the eigenvalues (or separation constants) k_x and k_y.

The general solution for $X(x)$ is

$$X(x) = A\cos k_x x + B\sin k_x x\,, \tag{3.25}$$

and we are specifying boundary conditions at $x = 0$ and $x = a$. In order to satisfy the boundary condition $X(0) = X(a) = 0$, we must have $A = 0$ and $k = n\pi/a$, leading to

$$X(x) = B\sin(k_n a) \qquad k_n = \frac{n\pi}{a} \quad n = 1, 2, \ldots\,. \tag{3.26}$$

Similarly

$$Y(y) = B\sin(k_m a) \qquad k_m = \frac{m\pi}{a} \quad m = 1, 2, \ldots \tag{3.27}$$

Thus the parallel directions determine both the functions and the separation constants. The complete eigen functions are

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad n = 1\ldots\infty \qquad m = 1\ldots\infty$$

(c) Finally we return to the perpendicular direction, Eq. (3.21). This equation does not usually constrain the separation constants. The general solution is

$$Z(z) = Ae^{k_z z} + Be^{-k_z z} \tag{3.28}$$

with $k_z = \sqrt{k_n^2 + k_m^2}$. With $Z(z)$ specified The general solution then is a linear combination

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}e^{-\gamma_{nm}z} + B_{nm}e^{+\gamma_{nm}z}\right]\psi_{nm}(x, y) \tag{3.29}$$

Solving the separated equations:

After separating variables, *all* of the equations we wil study can be written in Sturm Louiville form:

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] y(x) = \lambda r(x) y(x) \tag{3.30}$$

where $p(x)$ and $r(x)$ are postive definite fcns. Here we record some general properties of these equations.

(a) Given two independent solutions to the differential equation $y_1(x)$ a and $y_2(x)$ The wronskian times $p(x)$ is constant.

$$p(x)\underbrace{\left[y_1(x)y_2'(x) - y_2(x)y_1'(x)\right]}_{\text{wronskian(x)}} = const \tag{3.31}$$

This usually amounts to a statement of Gauss Law.

(b) If homogeneous boundary conditions are specified at two endpoints, $x = a$ and $x = b$, then the problem becomes an *eigenvalue* equation. Examples of the eigenfunctions we need are given in Appendix C.

In this case only certain values of $\lambda = \lambda_n$ are allowed and the functions are uniquely determined up to normalization

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] \psi_n(x) = \lambda_n r(x) \psi_n(x) \tag{3.32}$$

The parallel equations will have this form (see Eq. (3.19)), and notice how the boundary conditions at $x = 0$ and $x = a$ fixed the value of k_n (see Eq. (3.25) and Eq. (3.26)).

i) The resulting eigenfunctions are complete [3] and orthogonal with respect to the weight $r(x)$

$$\langle \psi_n, \psi_m \rangle = \int_a^b dx\, r(x) \psi_n^*(x) \psi_m(x) = C_n \delta_{nm} \tag{3.33}$$

where a and b are the endpoints where the boundary conditions are specified. The eigen-functions are usually not normalized.

ii) Completeness means that *any* function $f(x)$ satisfying the boundary conditions, can be expanded in the set

$$f(x) = \sum_n f_n \frac{1}{C_n} \psi_n(x)\,, \tag{3.34}$$

where the C_n are the normalization constants of the eigen-functions (Eq. (3.33)), and f_n is the inner product between $\psi_n(x)$ and $f(x)$ and $f(x)$

$$f_n = \langle \psi_n, f \rangle = \int_a^b dx\, r(x) \psi_n^*(x) f(x) \tag{3.35}$$

iii) One can easily show by substuting Eq. (3.35) into Eq. (3.34) that completeness implies

$$\sum_n \frac{\psi_n(x)\psi_n(x')}{C_n} = \frac{1}{r(x)}\delta(x - x') \tag{3.36}$$

(c) **S**olving the separated equations with δ function source terms

We will also need to know the green function of the one dimensional equation

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] g(x, x_o) = \delta(x - x_o) \tag{3.37}$$

[3] See Morse and Freshbach

The Green function for such 1D equations is based on knowing two homogeneous solutions $y_{out}(x)$ and $y_{\rm in}(x)$, where $y_{\rm out}(x)$ satisfies the boundary conditions for $x > x_o$, and $y_{\rm in}(x)$ satisfies the boundary conditions for $x < x_o$.

The Green function is continuous but has discontinuous derivatives. Since we know the solutions outside and inside it takes the form:

$$G(x, x_o) = C \left[y_{\rm out}(x) y_{\rm in}(x_o) \theta(x - x_o) + y_{\rm in}(x) y_{\rm out}(x_o) \theta(x_o - x) \right] \tag{3.38}$$

$$\equiv C y_{\rm out}(x_>) y_{\rm in}(x_<) \tag{3.39}$$

where C is a constant determined by integrating the equation, Eq. (3.37), across the delta function. In the second line we use the common (but somewhat confusing notation)

$$x_> \equiv \text{the greater of } x \text{ and } x_o \tag{3.40}$$

$$x_< \equiv \text{the smaller of } x \text{ and } x_o \tag{3.41}$$

which makes the second line mean the same as the first line.

Integrating from $x = x_o - \epsilon$ to $x = x_o + \epsilon$ we find the jump condition which enters in many problems:

$$-p(x) \frac{dg}{dx}\bigg|_{x_o+\epsilon} + p(x) \frac{dg}{dx}\bigg|_{x_o-\epsilon} = 1\,, \tag{3.42}$$

which can be used to find C.

(d) In fact the jump condition will always involve the Wronskian of the two solutions. Substituting Eq. (3.38) into Eq. (3.42) we see that $C = 1/(p(x_o)W(x_o))$

$$G(x, x_o) = \frac{\left[y_{\rm out}(x) y_{\rm in}(x_o) \theta(x - x_o) + y_{\rm in}(x) y_{\rm out}(x_o) \theta(x_o - x) \right]}{p(x_o)W(x_o)} \tag{3.43}$$

$$\equiv \frac{y_{\rm out}(x_>) y_{\rm in}(x_<)}{p(x_o)W(x_o)} \tag{3.44}$$

where $W(x_o) = y_{\rm out}(x_o) y'_{\rm in}(x_o) - y_{\rm in}(x_o) y'_{\rm out}(x_o)$ is the Wronskian. Note that the denominator $p(x_o)W(x_o)$ is constant and is independent of x_o.

4 Electric Fields in Matter

4.1 Parity and Time Reversal

(a) We discussed how fields transform under parity and time reversal. A useful table is

Quantity	Parity	Time Reversal
$\boldsymbol{r}(t)$	Odd	Even
$\boldsymbol{p}(t)$	Odd	Odd
$\mathrm{L} = \boldsymbol{r} \times \boldsymbol{p}$	Even	Odd
$\boldsymbol{F}$ =force	Odd	Even
$Q =$ charge	Even	Even
ρ	Even	Even
$\boldsymbol{j}$	Odd	Odd
$\boldsymbol{E}$	Odd	Even
$\boldsymbol{B}$	Even	Odd

(b) In the table above the force is odd if parity is a symmetry of the theory. Similarly $\boldsymbol{j}$ is odd under time reversal only if time-reversal is a symmetry of the theory. In a dissipative media, $\boldsymbol{j}$ is not odd under time-reversal (though the microscopic currents are) and time-reversal is not a symmetry of macroscopic electrodynamics.

(c) For example, for a parity invariant theory, a solution to the maxwell equations $\boldsymbol{E}(t,\boldsymbol{x}), \boldsymbol{B}(t,\boldsymbol{x})$ determines a new solution to the Maxwell equations $\underline{E}(t,\boldsymbol{x}), \underline{\boldsymbol{B}}(t,\boldsymbol{x})$ can be found through inversion

$$\boldsymbol{E}(t,\boldsymbol{x}) \rightarrow \underline{\boldsymbol{E}}(t,\boldsymbol{x}) = -\boldsymbol{E}(t,-\boldsymbol{x}) \tag{4.1}$$
$$\boldsymbol{B}(t,\boldsymbol{x}) \rightarrow \underline{\boldsymbol{B}}(t,\boldsymbol{x}) = \boldsymbol{E}(t,-\boldsymbol{x}) \tag{4.2}$$

as specified by last two rows of the first column of the table

4.2 Electrostatics in Material

Basic setup

(a) In material we expand the medium currents $\boldsymbol{j}_{\mathrm{m}at}$ in terms of a constitutive relation, fixing the currents in terms of the applied fields.

$$\boldsymbol{j}_{\mathrm{m}at} = [\text{ all possible combinations of the fields and their derivatives}] \tag{4.3}$$

We have added a subscript mat to indicate that the current is a medium current. There is also an external current $\boldsymbol{j}_{\mathrm{e}xt}$ and charge density $\rho_{\mathrm{e}xt}$.

(b) When only uniform electric fields are applied, and the electric field is weak, and the medium is isotropic, the polarization current takes the form

$$\boldsymbol{j}_{\text{mat}} = \sigma \boldsymbol{E} + \chi \partial_t \boldsymbol{E} + \dots \tag{4.4}$$

where the ellipses denote higher time derivatives of electric fields, which are suppressed by powers of t_{micro}/T_{macro} by dimensional analysis. For a conductor σ is non-zero. For a dielectric insulator σ is zero, and then the current takes the form

$$\boldsymbol{j}_b = \partial_t \boldsymbol{P} \tag{4.5}$$

- $\boldsymbol{P}$ is known as the polarization, and can be interpreted as the dipole moment per volume.
- We have worked with linear response for an isotropic medium where

$$\boldsymbol{P} = \chi \boldsymbol{E} \tag{4.6}$$

This is most often what we will assume.

For an anisotropic medium, χ is replaced by a susceptibility tensor

$$\boldsymbol{P}_i = \chi_{ij} \boldsymbol{E}^j \tag{4.7}$$

For a nonlinear (isotropic) medium $\boldsymbol{P}$ one could try a non-linear vector function of $\boldsymbol{E}$,

$$\boldsymbol{P}(\boldsymbol{E}) \tag{4.8}$$

defined by the low-frequency expansion of the current at zero wavenumber, but this is rather too simplistic for real ferro-electrics.

(c) Current conservation $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$ determines then that

$$\rho_{\text{mat}} = -\nabla \cdot \boldsymbol{P} \tag{4.9}$$

(d) The electrostatic maxwell equations read

$$\nabla \cdot \boldsymbol{E} = \underbrace{-\nabla \cdot \boldsymbol{P}}_{\rho_{mat}} + \rho_{\text{ext}} \tag{4.10}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{4.11}$$

or

$$\nabla \cdot \boldsymbol{D} = \rho_{ext} \tag{4.12}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{4.13}$$

where the *electric displacement* is

$$\boldsymbol{D} \equiv \boldsymbol{E} + \boldsymbol{P} \tag{4.14}$$

(e) For a linear isotropic medium

$$\boldsymbol{D} = (1 + \chi)\boldsymbol{E} \equiv \varepsilon \boldsymbol{E} \tag{4.15}$$

but in general $\boldsymbol{D}$ is a function of $\boldsymbol{E}$ which must be specified before problems can be solved.

Working problems with Dielectrics

(a) Using Eq. (4.9) and the Eq. (4.12) we find the boundary conditions that *normal* components of $\boldsymbol{D}$ jump across a surface if there is external charge, while the *parallel* components $\boldsymbol{E}$ are continuous

$$\boldsymbol{n}\cdot(\boldsymbol{D}_2-\boldsymbol{D}_1)=\sigma_{\rm ext} \qquad D_{2\perp}-D_{1\perp}=\sigma_{\rm ext} \tag{4.16}$$

$$\boldsymbol{n}\times(\boldsymbol{E}_2-\boldsymbol{E}_1)=0 \qquad E_{2\parallel}-E_{1\parallel}=0 \tag{4.17}$$

Very often $\sigma_{\rm ext}$ will be absent and then $D_\perp$ will be continuous (but *not* $E_\perp$).

(b) A jump in the polarization induces bound surface charge at the jump.

$$-\boldsymbol{n}\cdot(\boldsymbol{P}_2-\boldsymbol{P}_1)=\sigma_{\rm mat} \tag{4.18}$$

(c) Since the curl of $\boldsymbol{E}$ is zero we can always write

$$\boldsymbol{E}=-\nabla\varphi \tag{4.19}$$

and for linear media ($\boldsymbol{D}(\boldsymbol{r})=\varepsilon(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})$) with a non-constant dielectric constant $\varepsilon(\boldsymbol{r})$, we find an equation for $\boldsymbol{D}$

$$\nabla\cdot\varepsilon(\boldsymbol{r})\nabla\varphi=0 \tag{4.20}$$

(d) With the assumption of a linear medium $\boldsymbol{D}=\varepsilon\boldsymbol{E}$ and constant dielectric constant, the equations for electrostatics in medium are essentially identical to electrostatics without medium

$$-\varepsilon\nabla^2\Phi=\rho_{ext}\,, \tag{4.21}$$

but, the new boundary conditions lead to some (pretty minor) differences in the way the problems are solved.

Energy and Stress in Dielectrics:

(a) We worked out the extra energy stored in a dielectric as an ensemble of external charges are placed into the dielectric. As the macroscopic electric field $\boldsymbol{E}$ and displacement $\boldsymbol{D}(\boldsymbol{E})$ are changed by adding external charge $\delta\rho_{ext}$, the change in energy stored in the capacitor material is

$$\delta U=\int_V {\rm d}^3x\,\boldsymbol{E}\cdot\delta\boldsymbol{D} \tag{4.22}$$

(b) For a linear dielectric δU can be integrated, becoming

$$U=\tfrac{1}{2}\int_V {\rm d}^3x\,\boldsymbol{E}\cdot\boldsymbol{D}=\tfrac{1}{2}\int_V {\rm d}^3x\,\varepsilon\boldsymbol{E}^2 \tag{4.23}$$

(c) We worked out the stress tensor for a linear dielectric and found

$$T_E^{ij}=-\tfrac{1}{2}(D^iE^j+E^iD^j)+\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\delta^{ij} \tag{4.24}$$

$$=\varepsilon\left(-E^iE^j+\frac{1}{2}\boldsymbol{E}^2\delta^{ij}\right) \tag{4.25}$$

where in the first line we have written the stress in a form that can generalize to the non-linear case, and in the second line we used the linearity to write it in a form which is proportional the vacuum stress tensor.

(d) As always the force per volume in the Dielectric is

$$f^j = -\partial_i T_E^{ij} \tag{4.26}$$

where

$$T^{ij} = \text{the force in the } j\text{-th direction per area in the } i\text{-th} \tag{4.27}$$

More precisely let $\boldsymbol{n}$ be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, } \textit{by region LEFT on region RIGHT} \quad . \tag{4.28}$$

We can integrate the force/volume to find the net force on a given volume

$$F^j = \int_V d^3x\, f^j(\boldsymbol{x}) = -\int_{\partial V} da_i\, T^{ij} \tag{4.29}$$

This can be used to work out the force at a dielectric interface as done in lecture.

5 Ohms Law and Conduction

5.1 Steady current and Ohms Law:

(a) For steady currents

$$\nabla \cdot \boldsymbol{j} = 0 \tag{5.1}$$

(b) For steady currents in ohmic matter

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{5.2}$$

(c) σ has units of $1/s$. Note that in MKS units σ_{MKS} has the uninformative unit 1/ohm m:

$$\sigma_{HL} = \frac{\sigma_{MKS}}{\varepsilon_o} \tag{5.3}$$

For $\sigma_{MKS} = 10^7\,(\text{ohm m})^{-1}$ we find $\sigma \sim 10^{18}\,\text{s}^{-1}$.

(d) To find the flow of current we need to solve the electrostatics problem

$$-\nabla \cdot (\sigma \boldsymbol{E}) = 0 \tag{5.4}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{5.5}$$

or for homogeneous material ($\sigma = \text{const}$)

$$-\sigma \nabla^2 \Phi = 0 \tag{5.6}$$

We see that we are supposed to solve the Laplace equation. However the boundary conditions are rather different.

(e) A point source of current is represented by a delta function $I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$. While a sink of current is represented by a delta function of opposite sign $-I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$.

(f) Eq. (5.4) and Eq. (5.6) need boundary conditions. At an interface current should be conserved so

$$\boldsymbol{n} \cdot (\boldsymbol{j}_2 - \boldsymbol{j}_1) = 0 \tag{5.7}$$

or

$$\sigma_2 \frac{\partial \Phi_2}{\partial n} = \sigma_1 \frac{\partial \Phi_1}{\partial n} \tag{5.8}$$

Most often this is used to say that the normal component of the Electric field at a metal-insulator interface should be zero:

$$\boldsymbol{n} \cdot \boldsymbol{E} = 0 \qquad \text{at metal-insulator interface} \tag{5.9}$$

(g) In general the input current (or normal derivatives of the potential) must be specified at all the boundaries in order to have a well posed boundary value problem that can be solved (at least numerically.)

(h) In general the input currents $I_a = I_1, I_2, \ldots$ on a set conductors will be will be specified, specifying the normal derivatives on all of the surfaces. Then you solve for the potential. The voltages of a given electrode relative to ground is V_a , and you will find that $V_a = \sum_b R_{ab} I_b$. R_{ab} is the resistance matrix.

5.2 Basic physics of metals, Drude model of conductivity:

This section really lies outside of electrodynamics. But it helps to understand what is going on.

(a) The electrons in the metal under go scatterings with impurities and other defects on a time scale τ_c. For copper:

$$\tau_c \sim 10^{-14} s \tag{5.10}$$

(b) A typical coulomb oscillation / orbital frequency is set by the plasma frequency

$$\omega_p = \sqrt{\frac{ne^2}{m}} \tag{5.11}$$

For copper $\omega_{\boldsymbol{p}}$ is of order a typical quantum frequency and scales like:

$$\omega_p \sim \Big(\frac{1}{m} \underbrace{\frac{e^2}{a_o^3 m}}_{\text{spring const}} \Big)^{1/2} \tag{5.12}$$

$$\sim \left(\frac{27.2\,\text{eV}}{\hbar} \right) \tag{5.13}$$

$$\sim 10^{-16}\ 1/\text{s} \tag{5.14}$$

In the second to last line we ignored all 4π factors and used Bohr model identities

$$\frac{1}{2}\left(\frac{e^2}{4\pi a_o} \right) = \frac{\hbar^2}{2ma_o^2} = 13.6\,\text{e}V \tag{5.15}$$

which you can remember by noting that (minus) coulomb potential energy is twice the kinetic energy=$p^2/2m$ and knowing $p_{bohr} = \hbar/a_o$ as expected by the uncertainty principle.

(c) Since the distances between collisions are long compared to the Debroglie wavelength, and the time between collisions is long compared to a typical inverse quantum frequency, we are justified in using classical transport

$$\omega_p \tau_c \sim 100 \gg 1 \tag{5.16}$$

(d) In the Drude model the magnitude of the driving force $F_E = eE_{\text{ext}}$ equals the magnitude drag force $F_{\text{d}rag} = m\boldsymbol{v}/\tau_c$, leading to an estimate of the conductivity

$$\sigma = \frac{ne^2 \tau_c}{m} = \omega_p^2\, \tau_c \tag{5.17}$$

The estimates given show

$$\sigma \sim 10^{18}\ \text{s}^{-1} \tag{5.18}$$

for a metal like copper.

6 Magneto Statics and Magnetic Matter

6.1 Magneto-Statics

At first order in $1/c$ we have the magneto static equations

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}_{\rm tot}}{c} \qquad \boldsymbol{j}_{\rm tot} = \frac{\boldsymbol{j}}{c} + \underbrace{\frac{1}{c}\partial_t \boldsymbol{E}^{(0)}}_{\text{displacement current}} \tag{6.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.2}$$

where $\boldsymbol{j}_D = 1/c\,\partial_t \boldsymbol{E}^{(0)}$ is the displacement current. The formulas given below assume that $\boldsymbol{j}_D$ is zero. But, with no exceptions apply if one replaces $\boldsymbol{j} \to \boldsymbol{j} + \boldsymbol{j}_D$.

The current is taken to be steady

$$\nabla \cdot \boldsymbol{j} = 0 \tag{6.3}$$

Computing Fields:

(a) Below we note that for a current carrying wire

$$\boldsymbol{j}\,\mathrm{d}^3x = I\,\mathrm{d}\boldsymbol{\ell} \tag{6.4}$$

(b) We can compute the fields using the integral form of Ampères law $\nabla \times \boldsymbol{B} = j/c$, which says that the loop integral of $\boldsymbol{B}$ is equal to the current piercing the area bounded by the loop

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{I_{\rm pierce}}{c} \tag{6.5}$$

For the familiar case of a current carrying wire we found $B_\phi = (I/c)/2\pi\rho$, where ρ is the distance from the wire.

(c) The Biot-Savat Law is seemingly similar to the coulomb law

$$\boldsymbol{B}(\boldsymbol{r}) = \int \mathrm{d}^3 x_o \, \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c \times \widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{6.6}$$

We used this to compute the magnetic field of a ring of radius on the z-axis

$$B_z = 2\frac{(I/c)\pi a^2}{4\pi\sqrt{z^2 + a^2}} \tag{6.7}$$

which you can remember by knowing magnetic moment of the ring and other facts about magnetic dipoles (see below)

(d) Using the fact that $\nabla \cdot \boldsymbol{B} = 0$ we can write it as the curl of $\boldsymbol{A}$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda \tag{6.8}$$

but recognize that we can always add a gradient of a scalar function Λ to $\boldsymbol{A}$ without changing $\boldsymbol{B}$.

(e) If we adopt the coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and use the much used identity

$$\nabla \times (\nabla \times A) = -\nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{A}) \,, \tag{6.9}$$

we get the result

$$-\nabla^2 \boldsymbol{A} = \frac{\boldsymbol{j}}{c} \,. \tag{6.10}$$

Then in free space $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}(\boldsymbol{r}) = \int \mathrm{d}^3 x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{6.11}$$

(f) The equations must be supplemented by boundary conditions. In vacuum we have that the parallel components of $\boldsymbol{B}$ jump according to size of the surface currents $\boldsymbol{K}$, while the normal components of $\boldsymbol{B}$ are continuous

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}}{c} \tag{6.12}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{6.13}$$

Here $\boldsymbol{K}$ is the surface current and has units charge/length/s.

Multipole expansion of magnetic fields:

We wish to compute the magnetic field far from a localized set of currents. We can start with Eq. (6.14) and determine that far from the sources the vector potential is described by the magnetic dipole moment:

(a) The vector potential is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.14}$$

where

$$\boldsymbol{m} = \tfrac{1}{2} \int \mathrm{d}^3 x_o \boldsymbol{r}_o \times \boldsymbol{j}(\boldsymbol{r}_o)/c \tag{6.15}$$

is the magnetic dipole moment.

(b) For a current carrying wire:

$$\boldsymbol{m} = \frac{I}{c} \frac{1}{2} \oint \boldsymbol{r}_o \times \mathrm{d}\boldsymbol{\ell}_o = \frac{I}{c} \boldsymbol{a} \tag{6.16}$$

(c) The magnetic field from a dipole

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{3(\boldsymbol{n} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{6.17}$$

(d) **UNITS NOTE:** I defined $\boldsymbol{m}$ in Eq. (6.15) with $\boldsymbol{j}/c$. This has the “feature” that that

$$\boldsymbol{m}_{HL} = \frac{\boldsymbol{m}_{MKS}}{c} \tag{6.18}$$

In MKS units

$$\boldsymbol{A}_{MKS} = \mu_o \frac{\boldsymbol{m}_{MKS} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.19}$$

Setting $\varepsilon_o = 1$ so $\mu_o = 1/c^2$ and multiplying by c

$$\boldsymbol{A}_{HL} = c\boldsymbol{A}_{MKS} = \frac{\boldsymbol{m}_{MKS}/c \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\boldsymbol{m}_{HL} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.20}$$

Below we will define the magnetization, and similarly $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$.

Forces on currents

(a) We wish to compute the force on a small current carrying object in an external magnetic field. For a compact region of current (which is small compared to the inverse gradients of the external magnetic field) the total magnetic force is

$$\boldsymbol{F}(\boldsymbol{r}_o) = (\boldsymbol{m} \cdot \nabla)\, \boldsymbol{B}(\boldsymbol{r}_o) \tag{6.21}$$

where $\boldsymbol{m}$ is measured with respect $\boldsymbol{r}_o$, *i.e.*

$$\boldsymbol{m} = \frac{1}{2} \int_V \mathrm{d}^3x \; \delta\boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r})/c \tag{6.22}$$

with $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_o$.

(b) For a fixed dipole magnitude we have $\boldsymbol{F} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B})$ or

$$U(\boldsymbol{r}_o) = -\boldsymbol{m} \cdot \boldsymbol{B}(\boldsymbol{r}_o) \tag{6.23}$$

This formula is the same as the MKS one since we have taken $\boldsymbol{m}_{HL} = \boldsymbol{m}_{MKS}/c$.

(c) The torque is

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B} \tag{6.24}$$

(d) Finally (we will discuss this later) the magnetic force on a current carrying region is

$$(\boldsymbol{F}_B)^j = \frac{1}{c} \int_V (\boldsymbol{j} \times \boldsymbol{B})^j = -\int_{\partial V} dS \, \boldsymbol{n}_i T_B^{ij} \tag{6.25}$$

where

$$T_B^{ij} = -B^i B^j + \frac{1}{2} \boldsymbol{B}^2 \delta^{ij} \tag{6.26}$$

is the magnetic stress tensor and $\boldsymbol{n}$ is an outward directed normal.

Solving for magneto-static fields

(a) One approach is to use direct integration:

$$\boldsymbol{A}(\boldsymbol{r}) = \mu \int \mathrm{d}^3x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}$$

Then for any current distribution once can compute the magnetic field – see lecture for an example of a rotating charged sphere . This is analogous to using the coulomb law.

(b) Another approach is to view

$$-\nabla^2 \boldsymbol{A} = \mu \frac{\boldsymbol{j}}{c} \tag{6.27}$$

as a differential equation and to try separation of variables. There are (at least) two cases where the equations for $\boldsymbol{A}$ simplify.

i) If the current is azimuthally symmetric then it is reasonable to try a form $A_\phi(r, \theta)$

$$-\nabla^2 \boldsymbol{A} = \mu \frac{\boldsymbol{j}}{c} \Rightarrow -\nabla^2 A_\phi + \frac{A_\phi}{r^2 \sin^2\theta} = \mu \frac{j_\phi}{c} \tag{6.28}$$

Here the $-\nabla^2 A_\phi$ is the scalar Laplacian in spherical coordinates. For instance, this is an effective way to find the magnetic field from a ring of current or a rotating charged sphere. The appropriate separated solutions are given in Appendix E.2.

ii) If the current runs up and down then you can try $A_z(\rho, \phi)$ in cylindrical coordinates or 2D cartesian (x, y) coordinates

$$-\nabla^2 A_z(\rho, \phi) = \mu \frac{j_z}{c} \tag{6.29}$$

Here $\nabla^2 A_z$ is the scalar 2D Laplacian in cylindrical or cartesian coordinates, e.g. $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The separated solutions are identical to the 2D Laplace equation. See homework for an example of a cylindrical shell.

(c) Finally if the current separates two (or more) distinct regions of space (such as in a rotating charged sphere), then in each region one has

$$\nabla \times \boldsymbol{H} = 0 \tag{6.30}$$

So for each region one can introduce a scalar potential ψ_m such that

$$\boldsymbol{H} = -\nabla \psi_m \tag{6.31}$$

and (using $\nabla \cdot \boldsymbol{B} = 0$) show that

$$-\mu \nabla^2 \psi_m = 0 \tag{6.32}$$

assuming μ is constant. Then the Laplace equation is solved in each region using the separated homogeneous solutions introduced in electrostatics. The boundary conditions (Eq. (6.49)) are used to connect the scalar potential across regions. The boundary conditions are markedly different from the electrostatic case, and this leads to markedly different solutions. See lecture for an example of the magnetic moment induced by an external field.

6.2 Magnetic Matter

Basic equations

(a) We are considering materials in the presence of a magnetic field. We write $\boldsymbol{j}_{\rm mat}$ (the medium (material) currents) as an expansion in terms of the derivatives in the magnetic field. For weak fields, and an isotropic medium , the lowest term in the derivative expansion, for a parity and time-reversal invariant material is

$$\frac{\boldsymbol{j}_{\rm mat}}{c} = \chi_m^B \nabla \times \boldsymbol{B} \tag{6.33}$$

where we have inserted a factor of c for later convenience.

(b) The current takes the form

$$\frac{\boldsymbol{j}_{\rm mat}}{c} = \nabla \times \boldsymbol{M} \tag{6.34}$$

i) $\boldsymbol{M}$ is known as the magnetization, and can be interpreted as the magnetic dipole moment per volume.

ii) We have worked with linear response for an isotropic medium where

$$\boldsymbol{M} = \chi_m^B \boldsymbol{B} \tag{6.35}$$

This is most often what we will assume.

iii) Usually people work with $\boldsymbol{H}$ (see the next items (c), (d) for the definition of $\boldsymbol{H}$) not $\boldsymbol{B}$ [1]

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \tag{6.36}$$

[1] There are a couple of reasons for this. One reason is because the parallel components of H are continuous across the sample. But, ultimately it is $\boldsymbol{B}$ which is the curl $\boldsymbol{A}$, and it is ultimately the average current which responds to the gauge potential, through a retarded medium current-current correlation function that we wish to categorize.

iv) For not-that soft ferromagnets $\boldsymbol{M}(\boldsymbol{B})$ can be a very non-linear function of $\boldsymbol{B}$. This will need to be specified (usually by experiment) before any problems can be solved. Usually this is expressed as the magnetic field as a function of $\boldsymbol{H}$

$$\boldsymbol{B}(\boldsymbol{H}) \tag{6.37}$$

where $\boldsymbol{H}$ is small (of order gauss) and $\boldsymbol{B}$ is large (of order Tesla)

(c) After specifying the currents in matter, Maxwell equations take the form

$$\nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{M} + \frac{\boldsymbol{j}_{\rm ext}}{c} \tag{6.38}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.39}$$

or

$$\nabla \times \boldsymbol{H} = \frac{\boldsymbol{j}_{\rm ext}}{c} \tag{6.40}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.41}$$

where [2]

$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{M} \tag{6.43}$$

(d) For linear materials :

$$\boldsymbol{B} = \mu \boldsymbol{H} = \frac{1}{1-\chi_m^B} \boldsymbol{H} = (1+\chi_m)\boldsymbol{H} \tag{6.44}$$

Implying the definitions

$$\mu \equiv \frac{1}{1-\chi_m^B} \equiv (1+\chi_m) \tag{6.45}$$

Solving magnetostatic problems with linear magnetic media:

All of the methods described in Sect. (6.1) will work with minor modifications due to the boundary conditions described below

(a) For linear materials in the coulomb gauge we get

$$\nabla \times \boldsymbol{H} = \mu \frac{\boldsymbol{j}_{ext}}{c} \tag{6.46}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.47}$$

and with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and constant μ we find

$$-\nabla^2 \boldsymbol{A} = \mu \frac{j_{\rm ext}}{c} \tag{6.48}$$

which can be solved using the methods of magnetostatics.

(b) To solve magneto static equations we have boundary conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{\boldsymbol{K}_{\rm ext}}{c} \tag{6.49}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{6.50}$$

i.e. if there are no external currents then the parallel components of $\boldsymbol{H}$ are continuous and the perpendicular components of $\boldsymbol{B}$ are continuous.

[2] In the MKS system one has $\boldsymbol{H}_{MKS} = \frac{1}{\mu_o}\boldsymbol{B}_{MKS} - M_{MKS}$ so that $\boldsymbol{B}$ and $\boldsymbol{H}$ have different units. In a system of units where $\varepsilon_o = 1$ (so $1/\mu_o = c^2$) we have $H_{HL} = H_{MKS}/c$, $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$ or since $1/c = \sqrt{\mu_o}$:

$$\boldsymbol{H}_{HL} = \sqrt{\mu_o} H_{MKS} \qquad \boldsymbol{M}_{HL} = \sqrt{\mu_o}\boldsymbol{M}_{MKS} \tag{6.42}$$

(c) At an interface the there are bound currents which are generated

$$\boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) = \frac{\boldsymbol{K}_{\rm mat}}{c} \tag{6.51}$$

7 The Maxwell Equations and Quasistatic Fields

7.1 The Maxwell Equations a Summary

The maxwell equations in linear media can be written down for the gauge potentials. You should feel comfortable deriving all of these results directly from the Maxwell equations:

(a) The fields are

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{7.1}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{7.2}$$

(b) The equations of motion for the gauge potentials are in any gauge

$$-\Box\Phi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A}\right) = \rho \tag{7.3}$$

$$-\Box\boldsymbol{A} + \boldsymbol{\nabla}\left(\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A}\right) = \frac{\boldsymbol{j}}{c} \tag{7.4}$$

where the d'Alembertian is

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{7.5}$$

Note that these equations for Φ and $\boldsymbol{A}$ can not be solved without specifying a gauge constraint, *i.*e. given current conservation:

$$\partial_t\rho + \nabla\cdot\boldsymbol{j} = 0 \tag{7.6}$$

There are actually only three equations, but four unknowns.

(c) If the *coulomb gauge* is specified

$$\nabla\cdot\boldsymbol{A} = 0 \tag{7.7}$$

the equations read:

$$-\nabla^2\Phi = \rho \tag{7.8}$$

$$-\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t(-\nabla\Phi) \tag{7.9}$$

(d) If the *covariant gauge* is specified

$$\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A} = 0 \tag{7.10}$$

then the equations read

$$-\Box\Phi = \rho \tag{7.11}$$

$$-\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c} \tag{7.12}$$

8 Induction and Quasi-Static Fields

8.1 Induction and the energy in static Magnetic fields

(a) The Faraday law of induction says that changing magnetic flux induces an electric field

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{8.1}$$

In integral form

$$\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{c}\partial_t \Phi_B \qquad \Phi_B = \int_{\text{area}} \boldsymbol{B}\cdot \mathrm{d}\mathbf{a} = \oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{\ell} \tag{8.2}$$

(b) Faraday's Law is suppressed by $1/c^2$ relative to the coulomb law

(c) Faraday's law can be used to compute the energy stored in a magneto static field. As the currents are increased and the magnetic field is changed, the increase in energy stored in the magnetic fields and associated magnetization is

$$\delta U = \int_V \boldsymbol{H}\cdot \delta\boldsymbol{B}\, dV \tag{8.3}$$

For linear material $\boldsymbol{B} = \mu\boldsymbol{H}$

$$U = \frac{1}{2}\int \boldsymbol{H}\cdot\boldsymbol{B}\,\mathrm{d}^3x \tag{8.4}$$

$$= \frac{1}{2\mu}\int \boldsymbol{B}\cdot\boldsymbol{B}\,\mathrm{d}^3x \tag{8.5}$$

This can also be expressed in terms of $\boldsymbol{A}$:

$$\delta U = \int_V \frac{\boldsymbol{j}}{c}\cdot\delta\boldsymbol{A} \tag{8.6}$$

and for linear material:

$$U = \frac{1}{2}\int_V \frac{\boldsymbol{j}}{c}\cdot\boldsymbol{A} \tag{8.7}$$

The factor 1/2 arises because we are double counting the integral over the current in much the same way that a factor of 1/2 appears in $U = \frac{1}{2}\int_V \rho\Phi$

(d) Using the coulomb gauge result, for vector potential we show that the energy stored in a magnetic field is

$$U = \frac{\mu}{2}\int \mathrm{d}^3x\,\mathrm{d}^3x_o\,\frac{\boldsymbol{j}(\boldsymbol{r})/c\cdot\boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{8.8}$$

(e) For a set of current loops $I_a = I_1, I_2, \ldots$, we have $(\boldsymbol{j}/c)\,\mathrm{d}^3x = (I/c)\,\mathrm{d}\boldsymbol{\ell}$.

i) The energy integral Eq. (8.8) can be written

$$U = \frac{1}{2}\sum_{a,b} I_a M_{ab} I_b \tag{8.9}$$

Here $M_{aa} \equiv L_a$ is the self inductance, while M_{ab} is the mutual inductance. This is the circuit analog of Eq. (8.8).

ii) The magnetic flux through the $a-th$ loop is

$$\frac{\Phi_a}{c} = M_{ab} I_b \tag{8.10}$$

Here the magnetic flux through a given loop is

$$\Phi_a \equiv \int_{\rm a-loop} \boldsymbol{B}\cdot {\rm d}\boldsymbol{a} = \oint_{\rm a-loop} {\rm d}\boldsymbol{\ell}\cdot\boldsymbol{A} \tag{8.11}$$

and the magnetic energy can also be written

$$U = \frac{1}{2}\sum_a \frac{I_a}{c}\Phi_a \tag{8.12}$$

This is the circuit analog of Eq. (8.7).

iii) The change in the magnetic energy (for fixed geometry) is

$$\delta U = \sum_a \frac{I_a}{c}\delta\Phi_a \tag{8.13}$$

iv) The back emf in the a-th loop is (at fixed geometry) :

$$\mathcal{E}_a = -\frac{1}{c}\partial_t\Phi_a = -M_{ab}\frac{dI_b}{dt} \tag{8.14}$$

v) For a small change in flux $\delta\Phi_a$ and a small displacement of the loops $\delta\boldsymbol{R}_a$ (at fixed currents), the change in the magnetic energy

$$\delta U = \overbrace{\frac{I_a}{c}\delta\Phi_a}^{\delta W_{\rm batt}} + \delta W_{\rm mech} \tag{8.15}$$

where the first term is the work done by the battery (to keep the current constant inspite of the back emf induced by the changing flux), and $W_{\rm mech} = -\boldsymbol{F}_a\cdot\delta\boldsymbol{R}_a$ is the mechanical work done by the external force moving the loops, and $\boldsymbol{F}_a$ is force on the $a-$th loop. $U_B = \int_V B^2/(2\mu) = \frac{1}{2}I_a M_{ab} I_b$ is a property of the initial and final magnetic fields and is independent of how these fields are achieved. This combined with Eqs. 8.9,8.10 gives

$$\boldsymbol{F}_a\cdot\delta\boldsymbol{R}_a = +\frac{1}{2}I_a\,\delta M_{ab}\,I_b \tag{8.16}$$

8.2 Quasi-static fields

(a) We studied a prototypical problem of a charging a capacitor plates. The maxwell equations are categorized by an expansion in $1/c$, *i.e.* that the speed of light is fast compared to L/T the characteristic lengths L and times T. In this approximation the fields are determined instantaneously across space. Organizing the maxwell equations

$$\nabla \cdot \boldsymbol{E} = \rho \tag{8.17}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{8.18}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{8.19}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{8.20}$$

in powers of $1/c$ we have:

i) 0th order:

$$\nabla \cdot \boldsymbol{E}^{(0)} = \rho \qquad \nabla \times \boldsymbol{B}^{(0)} = 0 \tag{8.21}$$

$$\nabla \times \boldsymbol{E}^{(0)} = 0 \qquad \nabla \cdot \boldsymbol{B}^{(0)} = 0 \tag{8.22}$$

ii) 1st order:

$$\nabla \cdot \boldsymbol{E}^{(1)} = 0 \qquad \nabla \times \boldsymbol{B}^{(1)} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E}^{(0)} \tag{8.23}$$

$$\nabla \times \boldsymbol{E}^{(1)} = 0 \qquad \nabla \cdot \boldsymbol{B}^{(1)} = 0 \tag{8.24}$$

iii) 2nd order:

$$\nabla \cdot \boldsymbol{E}^{(2)} = 0 \qquad \nabla \times \boldsymbol{B}^{(2)} = 0 \tag{8.25}$$

$$\nabla \times \boldsymbol{E}^{(2)} = -\frac{1}{c}\partial_t \boldsymbol{B}^{(1)} \qquad \nabla \cdot \boldsymbol{B}^{(2)} = 0 \tag{8.26}$$

iv) Third order ...

$$\nabla \cdot \boldsymbol{E}^{(3)} = 0 \qquad \nabla \times \boldsymbol{B}^{(3)} = +\frac{1}{c}\partial_t \boldsymbol{E}^{(2)} \tag{8.27}$$

$$\nabla \times \boldsymbol{E}^{(3)} = -\frac{1}{c}\partial_t \boldsymbol{B}^{(2)} \qquad \nabla \cdot \boldsymbol{B}^{(3)} = 0 \tag{8.28}$$

Often time this goes beyond what is needed. Often at 3rd oder and beyond we will need to consider radiation at this order ..., since the fields do not (in general) decay faster than $1/r$ at infinity.

(b) In the quasi-static approximation we find a series of the following form:

$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(2)} + \ldots \tag{8.29}$$

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(3)} + \ldots \tag{8.30}$$

were $E^{(2)}$ is smaller than $\boldsymbol{E}^{(0)}$ by a factor of $(L/(cT))^2$. Similarly, $\boldsymbol{B}^{(3)}$ is are typically smaller than $\boldsymbol{B}^{(1)}$ (the leading B) by a factor $(L/(cT))^2$. If the material is ferromagnetic then μ can enhance the strength of $\boldsymbol{B}$ relative to the naive estimates.

Quasi-static approximation with gauge-potentials

(a) We often solve for the gauge potentials Φ and $\boldsymbol{A}$ (instead of $\boldsymbol{E}$ and $\boldsymbol{B}$) order by order in $1/c$ instead of $\boldsymbol{E}$ and $\boldsymbol{B}$ (see below). For example to second order in the Coulomb gauge we have

i) 0th order:

$$-\nabla^2\Phi = \rho \qquad \text{(actually all orders)} \tag{8.31}$$

ii) 1s order :

$$-\nabla^2\boldsymbol{A} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t(-\nabla\Phi) \tag{8.32}$$

This is sufficient to determine the electric and magnetic field to *second order*

$$\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} - \nabla\Phi \tag{8.33}$$

The covariant gauge can be studied similarly:

(b) In the covariant gauge we have

i) 0th:

$$-\nabla^2\Phi^{(0)} = \rho \tag{8.34}$$

ii) 1st:

$$-\nabla^2\boldsymbol{A} = \frac{\boldsymbol{j}}{c} \tag{8.35}$$

Together with gauge constraint:

$$\frac{1}{c}\partial_t\Phi^{(0)} + \nabla\cdot\boldsymbol{A} = 0 \tag{8.36}$$

iii) 2nd:

$$-\nabla^2\Phi^{(2)} = -\frac{1}{c^2}\frac{\partial^2\Phi^{(0)}}{\partial t^2} \tag{8.37}$$

8.3 Quasi-static approximation in metals and skin depth

(a) For the metals we derived a (quasi-static) diffusion equation for $\boldsymbol{B}$ by taking the curl of Amperes law and using Faraday's law

$$\nabla^2 \boldsymbol{B} = \frac{\sigma\mu}{c^2}\partial_t \boldsymbol{B} \tag{8.38}$$

You should feel comfortable deriving this. This shows the magnetic field diffuses in metal, with diffusion coefficient

$$D = \frac{c^2}{\mu\sigma}\,. \tag{8.39}$$

The diffusion coefficient has units $(distance)^2/time$ and is for copper, $D \sim \frac{1\,cm^2}{millisec}$

(b) Eq. (8.38) should be compared to the diffusion equation for a drop of dye in a cup of water:

$$D\nabla^2 n = \partial_t n\,. \tag{8.40}$$

A Gaussian drop of dye spreads out in time, and the mean squared width of the the drop increases in time as :

$$(\Delta x)^2 = 2D\Delta t \tag{8.41}$$

(c) If the RHS of Eq. (8.38) (the induced current) is small compared to the LHS, then we can neglect the induced currents and the magnetic field is unscreened by the induced currents. In this case, the characteristic lengths L we are considering are shorter than the skin depth:

$$\delta \equiv \sqrt{\frac{2c^2}{\sigma\mu\omega}} \tag{8.42}$$

On length scales larger than δ the magnetic field is damped by induced currents:

$$L \ll \delta \qquad \text{magnetic field unscreened} \tag{8.43}$$

$$L \gg \delta \qquad \text{magnetic field screened} \tag{8.44}$$

At fixed L this can also be expressed in term of frequency, *i.e.* if ω is less than $\omega_{ind} \equiv c^2/\sigma L^2$ then the magnetic field is not screened at length L, but if ω is greater than $\omega_{ind} \equiv c^2/\sigma L^2$, then the magnetic field is screened at length L.

9 The conservation theorems

9.1 Energy Conservation

(a) For energy to be conserved we expect that the total energy density (energy per volume) $u_{\rm tot}$ to obey a conservation law

$$\partial_t u_{\rm tot} + \partial_i S^i_{\rm tot} = 0 \tag{9.1}$$

where $S_{\rm tot}$ is the total energy flux.

(b) We divide the energy density into a mechanical energy density $u_{\rm mech}$ ($e.g.$ $dU = T\,dS - p\,dV$) and an electromagnetic energy density $u_{\rm em}$

$$u_{\rm tot} = u_{\rm mech} + u_{\rm em} \tag{9.2}$$

where

$$u_{\rm em} = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B} \tag{9.3}$$

(c) The energy flux $\boldsymbol{S}$ is also divided into a mechanical energy flux and an electromagnetic energy flux

$$\boldsymbol{S}_{\rm tot} = \boldsymbol{S}_{\rm mech} + \boldsymbol{S}_{\rm em} \tag{9.4}$$

where the mechanical energy flux comes from forces between the different mechanical subsystem and

$$\boldsymbol{S}_{\rm em} = c\,\boldsymbol{E}\times\boldsymbol{H} \tag{9.5}$$

(d) In this way for a mechanically isolated system $U = \int u\,{\rm d}V$

$$\frac{dU_{\rm mech}}{dt} + \frac{dU_{\rm em}}{dt} = -\int_{\partial V}\boldsymbol{S}\cdot d\boldsymbol{a} \tag{9.6}$$

(e) The starting point of this derivation is

$$\partial_t u_{\rm mech} + \partial_i S^i_{\rm mech} = \boldsymbol{j}\cdot\boldsymbol{E} \tag{9.7}$$

and showing that

$$\boldsymbol{j}\cdot\boldsymbol{E} = -\partial_t u_{\rm em} - \partial_i S^i_{\rm em} \tag{9.8}$$

9.2 Momentum Conservation

(a) For momentum to be conserved we expect that the total momentum per volume $\mathbf{g}_{\rm tot}$ satisfies a conservation law

$$\partial_t g^j + \partial_i T^{ij}_{\rm tot} = 0 \tag{9.9}$$

where T^{ij} is the total stress tensor

(b) We divide the momentum density into a mechanical momentum density $\mathbf{g}_{\rm mech}$ and an electromagnetic momentum density $\mathbf{g}_{\rm em}$

$$\mathbf{g}_{\rm tot} = \mathbf{g}_{\rm mech} + \mathbf{g}_{\rm em} \tag{9.10}$$

where the electromagnetic momentum density is

$$\mathbf{g}_{\rm em} = \boldsymbol{D} \times \boldsymbol{B} = \frac{\mu\varepsilon}{c^2}\boldsymbol{S}\,. \tag{9.11}$$

The last step is valid for simple matter and $\mu\varepsilon/c^2 = (n/c)^2$ where $n = \sqrt{\mu\varepsilon}$ is the index of refraction.

(c) The stress tensor $T^{ij}_{\rm tot}$ is also divided into a mechanical stress tensor $T^{ij}_{\rm mech}$ and an electromagnetic stress $T^{ij}_{\rm em}$

$$T^{ij}_{\rm tot} = T^{ij}_{\rm mech} + T^{ij}{}_{\rm em} \tag{9.12}$$

where the mechanical stress comes from the forces between the different mechanical subsystem and

$$T^{ij}_{\rm em} = \underbrace{-\tfrac{1}{2}(D^iE^j + D^jE^i) + \tfrac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\delta^{ij}}_{\text{electric stress}} + \underbrace{-\tfrac{1}{2}(H^iB^j + B^jH^i) + \tfrac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\delta^{ij}}_{\text{magnetic stress}} \tag{9.13}$$

$$= \underbrace{\varepsilon\left(-E^iE^j + \tfrac{1}{2}E^2\delta^{ij}\right)}_{\text{electric}} + \underbrace{\frac{1}{\mu}\left(-B^iB^j + \tfrac{1}{2}B^2\delta^{ij}\right)}_{\text{magnetic}} \tag{9.14}$$

(d) In this way for a mechanically isolated system the total momentum $\boldsymbol{P} = \int \boldsymbol{g}\,{\rm d}V$

$$\frac{dP^j_{\rm mech}}{dt} + \frac{dP^j_{\rm em}}{dt} = -\int_{\partial V} da\,\boldsymbol{n}_i T^{ij} \tag{9.15}$$

(e) The starting point of this derivation is

$$\partial_t g^j_{\rm mech} + \partial_i T^{ij}_{\rm mech} = \rho E^j + (\boldsymbol{j}/c \times \boldsymbol{B})^j \tag{9.16}$$

and showing that

$$\rho E^j + (\boldsymbol{j}/c \times \boldsymbol{B})^j = -\partial_t g^j_{\rm em} - \partial_i T^{ij}_{\rm em} \tag{9.17}$$

9.3 Angular momentum conservation

(a) Given the symmetry of stress tensor $T^{ij} = T^{ji}$ and the conservation law

$$\partial_t g^j_{\rm tot} + \partial_i T^{ij}_{\rm tot} = 0 \tag{9.18}$$

Then one can prove that angular momentum density satisfies a conservation law

$$\partial_t(\boldsymbol{r} \times \mathbf{g}_{\rm tot})_i + \partial_\ell(\epsilon_{ijk} r^j T^{k\ell}_{\rm tot}) = 0 \tag{9.19}$$

where the total angular momentum density is $\boldsymbol{r} \times \mathbf{g}_{\rm tot}$

(b) The angular momentum is divided into its mechanical and electromagnetic pieces. The electromagnetic piece is:

$$\boldsymbol{L}_{\rm em} = \int_V \boldsymbol{r} \times \mathbf{g}_{\rm em} \tag{9.20}$$

(c) For a mechanically isolated system we have

$$\frac{d}{dt}\left(\boldsymbol{L}_{\rm mech} + \boldsymbol{L}_{\rm em}\right)_i = \underbrace{-\int_{\partial V} da\,\boldsymbol{n}_\ell\, \epsilon_{ijk} r^j T^{k\ell}_{\rm em}}_{\text{em torque on the system}} \tag{9.21}$$

10 Waves

10.1 Plane waves and the Helmhotz Equation

(a) We look for solutions which have a particular (eigen)-frequency dependence ω_n, $\boldsymbol{E} = \boldsymbol{E}_n(\boldsymbol{x})e^{-i\omega_n t}$. This is very similar to the way that we look for particular energies in quantum mechanics, going from the time-dependent Schrödinger equation to the time-independent Schrödinger equation.

$$\nabla \cdot \boldsymbol{D}_n(\boldsymbol{x}) = 0 \tag{10.1}$$

$$\nabla \times \boldsymbol{H}_n(\boldsymbol{x}) = \frac{-i\omega_n \boldsymbol{D}(\boldsymbol{x})}{c} \tag{10.2}$$

$$\nabla \times \boldsymbol{B}_n(\boldsymbol{x}) = 0 \tag{10.3}$$

$$\nabla \times \boldsymbol{E}_n(\boldsymbol{x}) = \frac{i\omega_n \boldsymbol{B}(\boldsymbol{x})}{c} \tag{10.4}$$

From which we deduce the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_n^2\,\mu\varepsilon}{c^2}\right)\boldsymbol{E}_n = 0 \tag{10.5}$$

$$\left(\nabla^2 + \frac{\omega_n^2\,\mu\varepsilon}{c^2}\right)\boldsymbol{H}_n = 0 \tag{10.6}$$

which is an equation for the eigen-frequencies ω_n and the corresponding solutions $\boldsymbol{H}_n(\boldsymbol{x}), \boldsymbol{E}_n(\boldsymbol{x})$. It is important to emphasize that for a bounded system not all frequencies will be possible and still satisfy the boundary conditions.

The general solution is a superposition of these eigen modes,

$$\boldsymbol{E}(t, \boldsymbol{x}) = \sum_n C_n \boldsymbol{E}_n(\boldsymbol{x})e^{-i\omega_n t} \tag{10.7}$$

where the (complex) coefficients are adjusted to match the initial amplitude and time derivative of the wave. As in quantum mechanics the eigen functions, are of interest in their own right.

We will drop the n sub label on the wave-functions and eigen-frequencies below.

(b) If we restrict our wave functions to have the form $\boldsymbol{E}_n(\boldsymbol{r}) \equiv \boldsymbol{E}_{\boldsymbol{k}}(\boldsymbol{r})$

$$E_{\boldsymbol{k}}(\boldsymbol{r}) = \vec{\mathscr{E}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10.8}$$

$$\boldsymbol{B}_{\boldsymbol{k}}(\boldsymbol{r}) = \vec{\mathscr{B}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10.9}$$

then we get a condition on the frequency

$$k^2 = \frac{\omega^2\mu\varepsilon}{c^2} \qquad \text{or} \qquad \omega(k) = \frac{c}{\sqrt{\mu\varepsilon}}\,k \tag{10.10}$$

We have not assumed that $\vec{\mathscr{E}}$, $\vec{\mathscr{B}}$, or $\boldsymbol{k}$ are real.

(c) Examining Eq. (10.10) we see that that the plane waves propagate with speed

$$v_\phi = \frac{\omega}{k} = \frac{c}{n} \tag{10.11}$$

where we have defined the index of refraction

$$n = \sqrt{\mu\varepsilon} \tag{10.12}$$

(d) For every $\boldsymbol{k}$ we find from the Maxwell equations conditions on $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$:

$$\boldsymbol{k}\cdot\vec{\mathscr{B}} = 0 \tag{10.13}$$

$$\boldsymbol{k}\cdot\vec{\mathscr{E}} = 0 \tag{10.14}$$

and

$$\boldsymbol{k}\times\vec{\mathscr{E}} = \frac{\omega}{c}\vec{\mathscr{B}} \tag{10.15}$$

This last condition can be written

$$\frac{1}{Z}\hat{\boldsymbol{k}}\times\vec{\mathscr{E}} = \vec{\mathscr{H}} \qquad \text{or} \qquad n\hat{\boldsymbol{k}}\times\vec{\mathscr{E}} = \vec{\mathscr{B}} \tag{10.16}$$

where we defined[1] the *relative impedance* Z

$$Z \equiv \sqrt{\frac{\mu}{\varepsilon}} \tag{10.18}$$

and the index of refraction $n = \sqrt{\mu\epsilon}$

(e) **Linear Polarization:** For $\boldsymbol{k}$ real, we get two possible directions $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$. $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, where $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are orthogonal to $\hat{\boldsymbol{k}}$ and $\boldsymbol{\epsilon}_1\times\boldsymbol{\epsilon}_2 = \hat{\boldsymbol{k}}$

$$\vec{\mathscr{E}} = \mathscr{E}_1\boldsymbol{\epsilon}_1 + \mathscr{E}_2\boldsymbol{\epsilon}_2 \tag{10.19}$$

and

$$\vec{\mathscr{H}} = \mathscr{H}_1\boldsymbol{\epsilon}_2 + \mathscr{H}_2(-\boldsymbol{\epsilon}_1) \tag{10.20}$$

and as usual $\mathscr{H} = \mathscr{E}/Z$ or $\mathscr{B} = n\mathscr{E}$

(f) **Circular Polarization:** Instead of using $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ we can define the circular polarization vectors $\boldsymbol{\epsilon}_\pm$

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \tag{10.21}$$

For which + describes light which has positive helicity (circular polarization according to right hand rule), while − describes light with negative helicity (circular polarization opposite to right and rule).

(g) The general solution for the elctric field in vacuum is

$$\boldsymbol{E}(t,\boldsymbol{x}) = \sum_{s=\pm}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\mathscr{E}_s e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega_k t}\boldsymbol{\epsilon}_s \tag{10.22}$$

where $\omega_k = ck/n$

[1] We call this the relative impedance because when the relative impedenance Z_{HL} is expressed in terms of the MKS quantities, we have

$$Z_{HL} = \frac{\sqrt{\mu_{MKS}/\epsilon_{MKS}}}{\sqrt{\mu_o/\epsilon_o}} \tag{10.17}$$

$\sqrt{\mu_o/\epsilon_o} \simeq 376\,\text{ohm}$ is called the impedance of the vacuum and has units of ohms. But setting ϵ_o to 1 one sees that the "impedance of the vacuum" is just $1/c$. $[1/c] = s/m$ is the unit of resistance in HL units

(h) **Power and Energy Transport**

i) For a general wave satisfying the Helmholtz equation (i.e. sunusoidal) we have the time averaged poynting flux

$$\boldsymbol{S}_{\text{av}}(\boldsymbol{r}) = \frac{1}{2}\text{Re}\Big[c\,\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\Big] \tag{10.23}$$

ii) For a general wave satisfying the Helmholtz equation (i.e. sunusoidal) we have the time averaged energy density :

$$u_{\text{av}}(\boldsymbol{r}) = \frac{1}{2}\text{Re}\left[\frac{1}{2}\varepsilon \boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{2\mu}\boldsymbol{B}\cdot\boldsymbol{B}^*\right] \tag{10.24}$$

iii) For a plane wave we have

$$u_{\text{av}} = \tfrac{1}{2}\epsilon|\vec{\mathscr{E}}|^2 \tag{10.25}$$

$$\boldsymbol{S}_{\text{av}} = \frac{c}{Z}|\vec{\mathscr{E}}|^2\,\hat{\boldsymbol{k}} \tag{10.26}$$

$$= \frac{c}{n}\,u_{\text{av}}\,\hat{\boldsymbol{k}} \tag{10.27}$$

10.2 Reflection at interfaces

Reflection at a Dielectric

(a) We studied the reflection at a dielectric interface of in plane polarized waves (these are called TM or transverse magnetic waves), and of out of plane polarized waves (these are called TE or transverse electric waves).

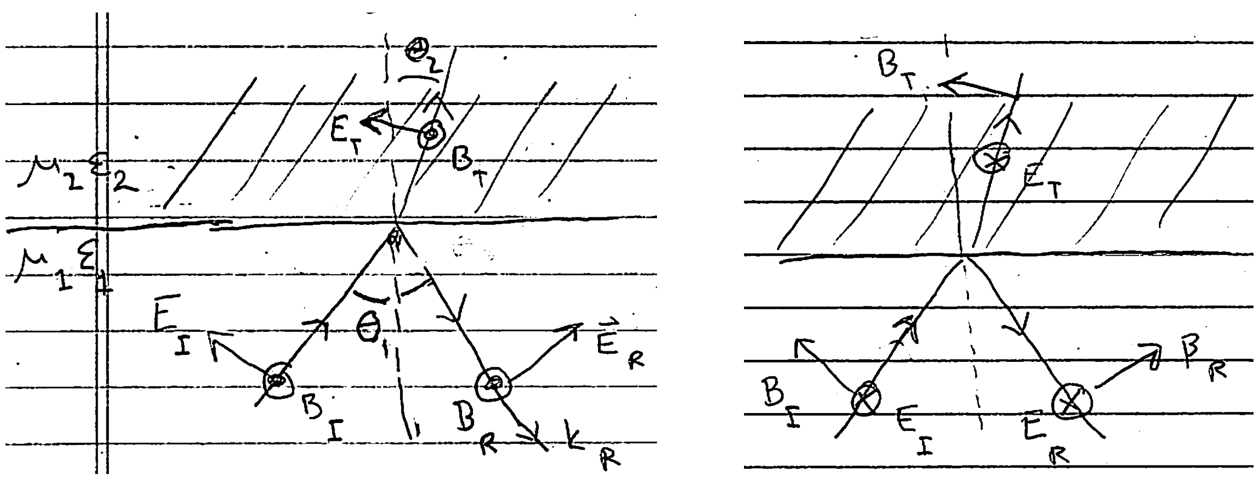


Figure 10.1: (a) Reflection of in plane polarized waves (transverse magnetic), and (b) Reflection of out of plane polarized waves (transverse electric)

(b) The waves in region 1 and region 2 are

$$\boldsymbol{E}_1 = \boldsymbol{E}_I e^{i\boldsymbol{k}_I\cdot\boldsymbol{r}-\omega t} + \boldsymbol{E}_R e^{i\boldsymbol{k}_R\cdot\boldsymbol{r}-\omega t} \tag{10.28}$$

$$\boldsymbol{E}_2 = \boldsymbol{E}_T e^{i\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t} \tag{10.29}$$

together with similar formulas for $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Note that $H = E/Z$

(c) By demanding the electromagnetic boundary conditions at the dielectric interface:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_2 - \boldsymbol{D}_1) = 0 \tag{10.30}$$

$$\boldsymbol{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0 \tag{10.31}$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{10.32}$$

$$\boldsymbol{n}\times(\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \tag{10.33}$$

we were able to conclude

i) Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{10.34}$$

ii) For *in plane polarized (TM=transverse magnetic) waves*:

$$\frac{E_R}{E_I} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2} \tag{10.35}$$

$$\frac{E_T}{E_I} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2} \tag{10.36}$$

where $Z = \sqrt{\mu/\epsilon}$, or $Z = 1/n$ when $\mu = 1$, and $\cos\theta_2 = \sqrt{1 - (n_1/n_2)^2 \sin^2\theta_1}$

iii) For *out of plane polarized (TE=transverse electric) waves*:

$$\frac{E_R}{E_I} = \frac{Z_2 \cos\theta_1 - Z_1 \cos\theta_2}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \tag{10.37}$$

$$\frac{E_T}{E_I} = \frac{2 Z_2 \cos\theta_1}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2} \tag{10.38}$$

iv) You should feel comfortable deriving these results.

(d) The reflection coefficient of in-plane (TM) waves vanishes at the Brewster angle $\tan\theta_B = n_1/n_2$. This means that upon reflection the light will be partially polarized.

Reflection at Metallic interface

(a) Compare the constituent relation for a metal and a dielectric:

$$\boldsymbol{j} = \sigma \boldsymbol{E} + \chi_e \partial_t \boldsymbol{E} + c\chi_m^B \, \nabla \times \boldsymbol{B} \qquad \text{Metal} \tag{10.39}$$

$$\boldsymbol{j} = \qquad \chi_e \partial_t \boldsymbol{E} + c\chi_m^B \, \nabla \times \boldsymbol{B} \qquad \text{Dielectric}\,, \tag{10.40}$$

in Fourier space

$$\boldsymbol{j} = -i\omega \boldsymbol{E}\left(\frac{i\sigma}{\omega} + \chi_e\right) + c\chi_m^B \, \nabla \times \boldsymbol{B} \qquad \text{Metal} \tag{10.41}$$

$$\boldsymbol{j} = -i\omega \boldsymbol{E}\left(\chi_e\right) \qquad c\chi_m^B \nabla \times \boldsymbol{B} \qquad \text{Dielectric}\,, \tag{10.42}$$

Thus (noting that $\varepsilon = 1 + \chi_e$) we see that the Maxwell equations in a metal merely involve the replacement $\chi_e \to \chi_e + i\sigma/\omega$, or

$$\varepsilon \to \hat{\varepsilon}(\omega) = \varepsilon + \frac{i\sigma}{\omega} \tag{10.43}$$

Usually $\sigma/\omega \gg \varepsilon$ and thus usually we replace:

$$\varepsilon \to \hat{\varepsilon}(\omega) \simeq \frac{i\sigma}{\omega} \tag{10.44}$$

(b) By looking for solutions of the form $\boldsymbol{H} = \boldsymbol{H}_c e^{i\boldsymbol{k}\cdot\boldsymbol{r} - \imath\omega t}$ in metal, we found $k_\pm^{\rm metal} = \pm(1+i)/\delta$, so for a wave propagating in the z direction the decaying amplitude is

$$\boldsymbol{H} = \boldsymbol{H}_c e^{ik_\pm^{\rm metal} z} = \boldsymbol{H}_c e^{iz/\delta} e^{-z/\delta} \tag{10.45}$$

we also found the (much smaller) electric field

$$\boldsymbol{E} = \sqrt{\frac{\mu\omega}{\sigma}} \frac{(1-i)}{\sqrt{2}} \boldsymbol{H}_c e^{iz/\delta} e^{-z/\delta} \tag{10.46}$$

which is suppressed by $\sqrt{\omega/\sigma}$ relative to $\boldsymbol{H}$

(c) We used these to study the reflection of light at a metal surface of high conductivity at normal incidence. This involves writing the fields outside the metal as a superposition of an ingoing and outgoing wave, and applying the boundary conditions as in the previous section to match the wave solutions across the interface. You should feel comfortable deriving these results.

(d) We analyzed the power flow in the reflection of light by the metal, and we analyzed the wave packet dynamics (see next section).

10.3 Waves in dielectrics and metals, dispersion

General Theory

(a) For maxwell equations at higher frequency the gradient expansion that we used should be replaced, as the frequency of the light is not small compared to atomic frequencies. However the wavelength λ is typically still much longer than the spacing between atoms, $\lambda \gg a_o$. Thus the expansion in spatial derivatives is still a good expansion. In a linear response approximation we write the current as an expansion:

$$\boldsymbol{j}(t,\boldsymbol{r}) = \int_{\infty}^{\infty} dt'\,\sigma(t-t')\boldsymbol{E}(t',\boldsymbol{r}) + \underbrace{\int dt' \chi_m^B(t-t')\, c\nabla \times B(t',\boldsymbol{r})}_{\text{often neglect}} \tag{10.47}$$

Often the magnetic response (which is smaller by $(v/c)^2$) is neglected.

(b) The functions are causal, we want them to vanish for $t' > t$, yielding

$$\sigma(t) = 0 \qquad t < 0 \tag{10.48}$$

$$\chi_m^B(t) = 0 \qquad t < 0 \tag{10.49}$$

(c) In frequency space the consituitent relation reads

$$\boldsymbol{j}(\omega,\boldsymbol{r}) = \sigma(\omega)\boldsymbol{E}(\omega,\boldsymbol{r}) + \underbrace{\chi_m^B(\omega)\, c\nabla \times \boldsymbol{B}(\omega,\boldsymbol{r})}_{\text{usually neglect}} \tag{10.50}$$

Motivated by considerations described below we will write the *same* function $\sigma(\omega)$ in a variety of ways

$$\sigma(\omega) \equiv -i\omega\chi_e(\omega) \quad \text{and} \quad \varepsilon(\omega) \equiv 1 + \chi_e(\omega) \equiv 1 + i\frac{\sigma(\omega)}{\omega} \tag{10.51}$$

(d) For low frequencies (less than an inverse collision timescale $\omega \ll 1/\tau_c$) our previous work applies. This this places constraints on $\sigma(\omega)$ at low frequencies

i) For a conductor for $\omega \ll \tau_c$, we need that $\boldsymbol{j}(t) = \sigma_o \boldsymbol{E}(t)$. This means that

$$\sigma(\omega) \simeq \sigma_o \quad \text{for} \quad \omega \to 0 \tag{10.52}$$

ii) For an insulator (dielectric) we had that $\boldsymbol{j}(t) = \partial_t \boldsymbol{P}(t) = \chi_e\,\partial_t \boldsymbol{E}$ so we expect that

$$\sigma(\omega) \simeq -i\omega\chi_e \quad \text{for} \quad \omega \to 0 \tag{10.53}$$

It is this different low frequency behavior of the conductivity that distinguishes a conductor from an insulator.

(e) With consituitive relation given in Eq. (10.50), and the continuity equation $-i\omega\rho(\omega) = -\nabla\cdot\boldsymbol{j}(\omega,\boldsymbol{r})$, we find that the maxwell equations in matter are formally the same as at low frequenency

$$\varepsilon(\omega)\nabla\cdot\boldsymbol{E}(\omega,\boldsymbol{r}) = 0 \tag{10.54}$$

$$\nabla\times B(\omega,\boldsymbol{r}) = \frac{-i\omega\varepsilon(\omega)\mu(\omega)}{c}\boldsymbol{E}(\omega) \tag{10.55}$$

$$\nabla\cdot\boldsymbol{B}(\omega,\boldsymbol{r}) = 0 \tag{10.56}$$

$$\nabla\times\boldsymbol{E}(\omega,\boldsymbol{r}) = \frac{i\omega}{c}\boldsymbol{B}(\omega,\boldsymbol{r}) \tag{10.57}$$

$\varepsilon(\omega)$ and $\mu(\omega)$ are complex functions of ω

$$\varepsilon(\omega) = 1 + \chi_e(\omega) \tag{10.58}$$

$$\mu(\omega) = \frac{1}{1-\chi_m^B(\omega)} \tag{10.59}$$

We gave two models for what $\varepsilon(\omega)$ might look like in dielectrics and metals (see below).

(f) Given the Maxwell equations we studied the propagation of transverse waves

$$\boldsymbol{E}_T(t,\boldsymbol{r}) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{x} - \imath\omega t} \tag{10.60}$$

with $\boldsymbol{E}_o \cdot \boldsymbol{k} = 0$. The helmholtz equation for transverse waves becomes:

$$\left[-k^2 + \frac{\omega^2 \varepsilon(\omega)\mu(\omega)}{c^2}\right]\boldsymbol{E}_o = 0\,. \tag{10.61}$$

where $\varepsilon(\omega) \equiv \varepsilon'(\omega) + i\varepsilon''(\omega)$ and $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$ are complex functions of frequency, with real parts, $\epsilon'(\omega), \mu'(\omega)$, and imaginary parts, $\epsilon''(\omega), \mu''(\omega)$. In general Eq. (10.61) determines to a relation between $\omega(k)$ and k for any specified $\epsilon(\omega)$ and $\mu(\omega)$. Usually we will set $\mu(\omega) = 1$.

(g) The real part of $\omega(k)$ is known as the dispersion curve and determines the phase and group velocities of the wave and wave packets. This is determined by the real part of the permitivity $\varepsilon(\omega)$. The imaginary part of the $\varepsilon(\omega)$ dtermines the absorption of the wave.

To see this we solved Eq. (10.61) with $\mu(\omega) = 1$ and the imaginary part of $\epsilon(\omega)$ small. Defining

$$\omega(k) \equiv \omega_o(k) - \tfrac{i}{2}\Gamma(k)\,, \tag{10.62}$$

so that

$$\boldsymbol{E}_T(t,\boldsymbol{x}) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_o(k)t} e^{-\frac{1}{2}\Gamma(k)t} \tag{10.63}$$

we find that $\omega_o(k)$ (which is known as the dispersion curve is) satisfies

$$-k^2 + \frac{\omega_o^2}{c^2}\epsilon'(\omega_o) = 0 \tag{10.64}$$

and the damping rate is

$$\Gamma(k) = \frac{\omega_o(k)\epsilon''(\omega_o(k))}{\epsilon'(\omega_o(k))} \tag{10.65}$$

(h) Sometimes it is easier to think about it as k as function of ω rather than $\omega(k)$. Solving Eq. (10.61) for k

$$k = \frac{\omega}{c} n(\omega)\,, \tag{10.66}$$

with $n(\omega) = \sqrt{\epsilon(\omega)}$, we find the wave form:

$$\boldsymbol{E}(t,\boldsymbol{r}) = \boldsymbol{E}_o e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{E}_o\, e^{-i\omega t}\, e^{i\frac{\omega n_1(\omega)}{c}z} e^{-\frac{\omega n_2(\omega)}{c}z} \tag{10.67}$$

where $n_1(\omega)$ is the real part of $n(\omega)$, and $n_2(\omega)$ is the imaginary part of $n(\omega)$.

Thus the real part of $n(\omega)$ determines the real wave number of the wave, $(\omega/c)\, n_1(\omega)$, while the imaginary part of $n(\omega)$, $n_2(\omega)$, determines the absorption of the wave as it propagates through media.

A model $\varepsilon(\omega)$ function for dielectrics

In general one needs to know how the medium reacts in order to determine $\sigma(\omega)$. At low frequency $\sigma(\omega)$ is determined by a few constants which are given by the taylor expansion of $\sigma(\omega)$. At higher frequency a detailed micro-theory is needed to compute $\sigma(\omega)$. The following model capture the qualitative features of dielectrics as a function of frequency. Replacing the model for a dielectric, with a quantum mechanical description of electronic oscillations in atoms, gives a realistic description of neutral gasses.

(a) For an insulator we gave a simple model for the dielectric, where the electrons are harmonically bound to the atoms. The equation of motion satisfied by the electrons are

$$m\frac{d^2\boldsymbol{x}}{dt^2} + m\eta\frac{d\boldsymbol{x}}{dt} + m\omega_o^2\boldsymbol{x} = e\boldsymbol{E}_{\rm ext}(t) \tag{10.68}$$

Solving for the current $\boldsymbol{j}(t) = \boldsymbol{j}_\omega e^{-i\omega t}$, with a sinusoidal field $\boldsymbol{E}(t) = E_\omega e^{i\omega t}$ we found $\chi_e(\omega)$

$$\varepsilon(\omega) = 1 + \chi_e(\omega) = 1 + \frac{\omega_p^2}{-\omega^2 + \omega_o^2 - i\omega\eta} \tag{10.69}$$

where the plasma frequency is

$$\omega_p^2 = \frac{ne^2}{m} \tag{10.70}$$

and at low frequency we recover Eq. (**??**)

$$\chi_e \simeq \frac{\omega_p^2}{\omega_o^2} \quad \text{for} \quad \omega \to 0 \tag{10.71}$$

10.4 Dynamics of wave packets

(a) Any real wave is a superposition of plane waves:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx-\omega(k)t} \tag{10.72}$$

The complex values of $A(k)$ can be adjusted so that at time $t = 0$ the initial conditions, $u(x, 0)$ and $\partial_t u(x, 0)$, can be satisfied.

(b) A proto-typical wave packet at time $t = 0$ is a Gaussian packet

$$u(x,0) = e^{ik_o x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-x_o)^2}{2\sigma^2}} \tag{10.73}$$

The spatial width is

$$\Delta x = \frac{\sigma}{\sqrt{2}} \tag{10.74}$$

The Fourier transform is

$$A(k) = \exp(-\tfrac{1}{2}(k-k_o)^2\sigma^2) \tag{10.75}$$

The wavenumber width

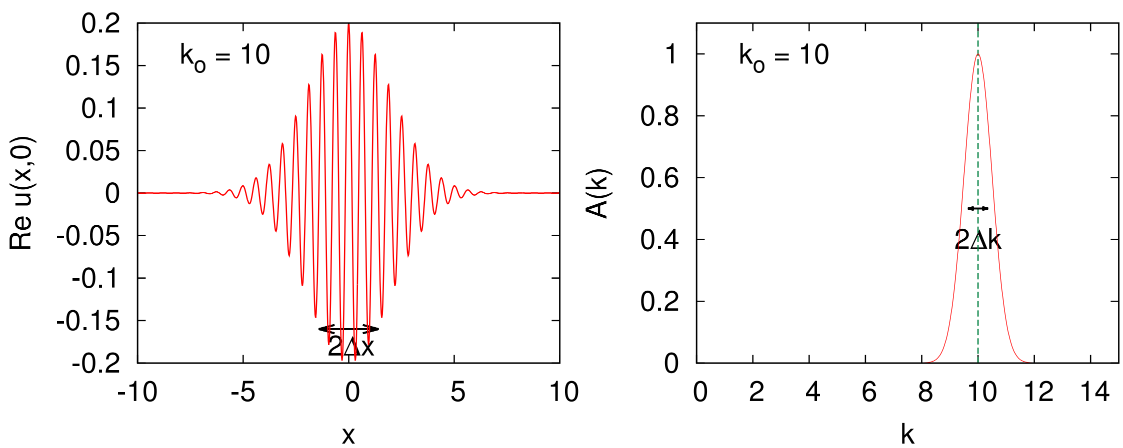
$$\Delta k = \frac{1}{\sqrt{2}\sigma} \tag{10.76}$$

so

$$\Delta k \Delta x = \frac{1}{2} \tag{10.77}$$

which saturates the uncertainty bound $\Delta x \Delta k \geq \frac{1}{2}$. The Gaussian is the unique wave form which saturates the bound.

A picture of these Fourier Transforms is



(c) The uncertainty relation relates the wavenumber and spatial widths

$$\Delta x \Delta k \geq \tfrac{1}{2} \tag{10.78}$$

where

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} |u(x,0)|^2 (x-\bar{x})^2}{\int_{\infty}^{\infty} |u(x,0)|^2} \tag{10.79}$$

$$(\Delta k)^2 = \frac{\int_{-\infty}^{\infty} |A(k)|^2 (k-\bar{k})^2}{\int_{\infty}^{\infty} |A(k)|^2} \tag{10.80}$$

(d) You should be able to derive that the center of the wave packet moves with the group velocity

$$v_g = \frac{d\omega}{dk} \tag{10.81}$$

In a very similar way one derives that, if a wave experiences a frequency dependent phase shift $\phi(\omega)$ upon reflection or transmission, the wave packet will be delayed relative to a geometric optics approximation by a time delay

$$\Delta = \frac{d\phi(\omega)}{d\omega} \tag{10.82}$$

11 Radiation in Non-relativistic Systems

11.1 Basic equations

This first section will *NOT* make a non-relativistic approximation, but will examine the far field limit.

(a) We wrote down the wave equations in the covariant gauge:

$$-\Box\Phi = \rho(t_o, \boldsymbol{r}_o) \tag{11.1}$$

$$-\Box\boldsymbol{A} = \boldsymbol{J}(t_o, \boldsymbol{r}_o)/c \tag{11.2}$$

The gauge condition reads

$$\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A} = 0 \tag{11.3}$$

(b) Then we used the green function of the wave equation

$$G(t, r|t_o r_o) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t - t_o + \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c}) \tag{11.4}$$

to determine the potentials $(\Phi, \boldsymbol{A})$

$$\Phi(t, \boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\rho(T, \boldsymbol{r}_o) \tag{11.5}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{11.6}$$

Here $T(t, \boldsymbol{r})$ is the retarded time

$$T(t, r) = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c} \tag{11.7}$$

(c) We used the potentials to determine the electric and magnetic fields. Electric and magnetic fields in the far field are

$$\boldsymbol{A}_{\mathrm{rad}}(t, \boldsymbol{r}) = \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{\boldsymbol{J}(T, \boldsymbol{r}_o)}{c} \tag{11.8}$$

and

$$\boldsymbol{B}(t, \boldsymbol{r}) = -\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} \tag{11.9}$$

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{n}\times\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} = -\boldsymbol{n}\times\boldsymbol{B}(t, \boldsymbol{r}) \tag{11.10}$$

In the far field (large distance limit $\boldsymbol{r}\to\infty$) limit we have

$$T = t - \frac{r}{c} + \boldsymbol{n}\cdot\frac{\boldsymbol{r}_o}{c} \tag{11.11}$$

And we recording the derivatives

$$\left(\frac{\partial}{\partial t}\right)_{r_o} = \left(\frac{\partial}{\partial T}\right)_{r_o} \tag{11.12}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T + \frac{\boldsymbol{n}}{c}\left(\frac{\partial}{\partial T}\right)_{\boldsymbol{r}_o} \tag{11.13}$$

(d) We see that the radiation (electric field) is proportional to the transverse piece of the $\partial_t \boldsymbol{J}$

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \partial_t \boldsymbol{J}) = \partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n} \cdot \partial_t \boldsymbol{J}) \tag{11.14}$$

In general the transverse projection of a vector is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{V}) = \boldsymbol{V} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{V}) \tag{11.15}$$

(e) Power radiated per solid angle is for $r \to \infty$ is

$$\frac{dW}{dtd\Omega} = \frac{dP(t)}{d\Omega} = \text{energy per observation time per solid angle} \tag{11.16}$$

and

$$\frac{dP(t)}{d\Omega} = r^2 \boldsymbol{S} \cdot n \tag{11.17}$$

$$= c|rE|^2 \tag{11.18}$$

11.2 Examples of Non-relativistic Radiation: L31

In this section we will derive several examples of radiation in non-relativistic systems. In a non-relativistic approximation

$$T = t - \frac{r}{c} + \underbrace{\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o}_{\text{small}} \tag{11.19}$$

The underlined terms are small: If the typical time and size scales of the source are $T_{\rm typ}$ and $L_{\rm typ}$, then $t \sim T_{\rm typ}$, and $\boldsymbol{r}_o \sim L_{\rm typ}$, and the ratio the underlined term to the leading term is:

$$\frac{L_{\rm typ}}{cT_{\rm typ}} \ll 1 \tag{11.20}$$

This is the non-relativistic approximation. For a harmonic time dependence, $1/T_{\rm typ} \sim \omega_{\rm typ}$, and this says that the wave number $k = \frac{2\pi}{\lambda}$ is small compared to the size of the source, *i.e. the wave length of the emitted light is long compared to the size of the system in non-relativistic motion:*

$$\frac{2\pi L_{\rm typ}}{\lambda} \ll 1 \tag{11.21}$$

(a) Keeping only $t - r/c$ and dropping all powers of $\boldsymbol{n} \cdot \boldsymbol{r}_o/c$ in T results in the electric dipole approximation, and also the Larmour formula.

(b) Keeping the first order terms in

$$\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o \tag{11.22}$$

results in the magnetic dipole and quadrupole approximations.

The Larmour Formula

(a) For a particle moves slowly with velocity and acceleration, $\boldsymbol{v}(t)$ and $\boldsymbol{a}(t)$ along a trajectory $\boldsymbol{r}_*(t)$

(b) We make an ultimate non-relativistic approximation for T

$$T \simeq t - \frac{r}{c} \equiv t_e \tag{11.23}$$

Then we derived the radiation field by substituting the current

$$\boldsymbol{J}(t_e) = e\boldsymbol{v}(t_e)\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_e)) \tag{11.24}$$

into the Eqs. (11.8),(11.9), and (11.17) for the radiated power

(c) The electric field is

$$\boldsymbol{E} = \frac{e}{4\pi r c^2}\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{a}(t_e) \tag{11.25}$$

Notice that the electric field is of order

$$E \sim \frac{e}{4\pi r}\,\frac{a(t_e)}{c^2} \tag{11.26}$$

(d) The power per solid angle emitted by acceleration at time t_e is

$$\frac{dP(t_e)}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3} a^2(t_e)\sin^2\theta \tag{11.27}$$

Notice that the power is of order

$$P \sim c|rE|^2 \sim \frac{a^2}{c^3} \tag{11.28}$$

(e) The total energy that is emitted is

$$P(t_e) = \frac{e^2}{4\pi}\frac{2}{3}\frac{a^2(t_e)}{c^3} \tag{11.29}$$

The Electric Dipole approximation

(a) We make the ultimate non-relativistic approximation

$$\boldsymbol{J}(t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}) \simeq \boldsymbol{J}(t - \frac{r}{c}) \tag{11.30}$$

Leading to an expression for $\boldsymbol{A}_{\rm rad}$

$$\boldsymbol{A}_{\rm rad} = \frac{1}{4\pi r}\frac{1}{c}\partial_t \boldsymbol{p}(t_e) \tag{11.31}$$

where the dipole moment is

$$\boldsymbol{p}(t_e) = \int \mathrm{d}^3 x_o\, \rho(t_e)\boldsymbol{r}_o \tag{11.32}$$

(b) The electric and magnetic fields are

$$\boldsymbol{E}_{\rm rad} = \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{c}\partial_t\boldsymbol{A}_{\rm rad} \tag{11.33}$$

$$= \frac{1}{4\pi r c^2}\,\boldsymbol{n}\times\boldsymbol{n}\times\ddot{\boldsymbol{p}}(t_e) \tag{11.34}$$

$$\boldsymbol{B}_{\rm rad} = \boldsymbol{n}\times\boldsymbol{E}_{\rm rad} \tag{11.35}$$

(c) The power radiated is

$$\frac{dP(t_e)}{d\Omega} = \frac{1}{16\pi^2}\frac{\ddot{p}^2(t_e)}{c^3}\sin^2\theta \tag{11.36}$$

(d) For a harmonic source $\boldsymbol{p}(t_e) = \boldsymbol{p}_o e^{-i\omega(t-r/c)}$ the time averaged power is

$$P = \frac{1}{4\pi}\frac{\omega^4}{3c^3}|\boldsymbol{p}_o|^2 \tag{11.37}$$

The magnetic dipole and quadrupole approximation: L32

(a) In the magnetic dipole and quadrupole approximation we expand the current

$$\boldsymbol{J}(T) \simeq \underbrace{\boldsymbol{J}(t_e)}_{\text{electric dipole}} + \underbrace{\frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}\partial_t \boldsymbol{J}(t_e, \boldsymbol{r}_o)/c}_{\text{next term}} \tag{11.38}$$

The next term when substituted into Eq. (11.8) gives rise two new contributions to $\boldsymbol{A}_{\rm rad}$, the magnetic dipole and electric quadrupole terms:

$$\boldsymbol{A}_{\rm rad} = \underbrace{\boldsymbol{A}^{E1}_{\rm rad}}_{\text{electric dipole}} + \underbrace{\boldsymbol{A}^{M1}_{\rm rad}}_{\text{mag dipole}} + \underbrace{\boldsymbol{A}^{E2}_{\rm rad}}_{\text{electric-quad}} \tag{11.39}$$

(b) The magnetic dipole contribution gives

$$\boldsymbol{A}^{M1}_{\rm rad} = \frac{-1}{4\pi r}\frac{\boldsymbol{n}}{c} \times \dot{\boldsymbol{m}}(t_e) \tag{11.40}$$

where $\boldsymbol{m}$

$$\boldsymbol{m} \equiv \frac{1}{2}\int_{\boldsymbol{r}_o} \boldsymbol{r}_o \times \boldsymbol{J}(t_e, \boldsymbol{r}_o)/c\,, \tag{11.41}$$

is the magnetic dipole moment.

(c) The structure of magnetic dipole radiation is very similar to electric dipole radiation with the duality transformation

$$\text{E-dipole} \to \text{M-dipole} \tag{11.42}$$
$$\boldsymbol{p} \to \boldsymbol{m} \tag{11.43}$$
$$\boldsymbol{E} \to \boldsymbol{B} \tag{11.44}$$
$$\boldsymbol{B} \to -\boldsymbol{E} \tag{11.45}$$

(d) The power is

$$\frac{dP^{M1}(t_e)}{d\Omega} = \frac{\ddot{m}^2 \sin^2\theta}{16\pi^2 c^3} \tag{11.46}$$

(e) The power radiated in magnetic dipole radiation is smaller than the power radiated in electric dipole radiation by a factor of the typical velocity, $v_{\rm typ}$ squared:

$$\frac{P^{M1}}{P^{E1}} \propto \frac{m^2}{p^2} \sim \left(\frac{v_{\rm typ}}{c}\right)^2 \tag{11.47}$$

where $v_{\rm typ} \sim L_{\rm typ}/T_{\rm typ}$

Quadrupole rdiation

(a) For quadrupole radiation we have

$$\boldsymbol{A}^{j}_{\rm rad,E2} = \frac{1}{24\pi r}\frac{n_i}{c^2}\dddot{\mathcal{Q}}^{ij} \tag{11.48}$$

where $\mathcal{Q}^{ij}$ is the symmetric traceless quadrupole tensor.

$$\mathcal{Q}^{ij} = \int d^3x_o \rho(t_e, \boldsymbol{r}_o)\,\left(3r_o^i r_o^j - \boldsymbol{r}_o^2\delta^{ij}\right) \tag{11.49}$$

(b) The electric field is

$$\boldsymbol{E}_{\rm rad} = \frac{-1}{24\pi r c^3}\left[\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^\top\cdot\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n})\right] \tag{11.50}$$

where (more precisely) the first term in square brackets means $n_i\dddot{\mathcal{Q}}^{ij}$, while the second term means, $(n_\ell \dddot{\mathcal{Q}}^{\ell m} n_m)n^j$.

(c) A fair bit of algebra shows that the total power radiated from a quadrupole form is

$$P = \frac{1}{720\pi c^5} \dddot{\mathcal{Q}}^{ab} \dddot{\mathcal{Q}}_{ab} \tag{11.51}$$

(d) For harmonic fields, $\mathcal{Q} = \mathcal{Q}_o e^{-i\omega t}$, the time averaged power is rises as ω^6

$$P = \frac{c}{1440\pi} \left(\frac{\omega}{c}\right)^6 \mathcal{Q}_o^2 \tag{11.52}$$

(e) The total power radiated radiated in quadrupole radiation to electric-dipole radiation for a typical source size $L_{\rm typ}$ is smaller:

$$\frac{P^{E2}}{P^{E1}} \sim \left(\frac{\omega L_{\rm typ}}{c}\right)^2 \tag{11.53}$$

11.3 Transition to the radiation zone

(a) Starting from the general expression Eq. (11.5), we studied the exact fields of an electric dipole. The current for the dipole is

$$\boldsymbol{J}(t_o, \boldsymbol{r}_o) = \partial_{t_o} \boldsymbol{p}(t_o) \delta^3(\boldsymbol{r}_o) \tag{11.54}$$

$$\boldsymbol{\rho}(t_o, \boldsymbol{r}_o) = -\boldsymbol{p}(t_o) \cdot \nabla_{\boldsymbol{r}_0} \delta^3(\boldsymbol{r}_o) \tag{11.55}$$

Performing the integrals in Eq. (11.5), and differentiating to find the electric and magnetic fields we have

$$\boldsymbol{E}(t, \boldsymbol{r}) = \underbrace{\frac{3(\boldsymbol{n} \cdot \boldsymbol{p}(t_e)) - \boldsymbol{p}}{4\pi r^3}}_{\text{near field}} + \underbrace{\frac{3\boldsymbol{n}(\boldsymbol{n} \cdot \dot{\boldsymbol{p}}(t_e)) - \dot{\boldsymbol{p}}(t_e)}{4\pi r^2 c}}_{\text{intermediate zone}} + \underbrace{\frac{-\ddot{\boldsymbol{p}}(t_e) + \boldsymbol{n}(\boldsymbol{n} \cdot \ddot{\boldsymbol{p}}(t_e))}{4\pi r c^2}}_{\text{radiation field} \propto \, \boldsymbol{n} \times \boldsymbol{n} \times \ddot{\boldsymbol{p}}} \tag{11.56}$$

(11.57)

and

$$\boldsymbol{B}(t, \boldsymbol{r}) = \underbrace{-\frac{\boldsymbol{n} \times \dot{\boldsymbol{p}}(t_e)}{4\pi r^2 c}}_{\text{quasi-static filed}} + \underbrace{-\frac{\boldsymbol{n} \times \ddot{\boldsymbol{p}}(t_e)}{4\pi r c^2}}_{\text{radiation field}} \tag{11.58}$$

(b) The successive terms trade powers of $1/r$ for powers of $1/c\,\partial_t$. The radiation field decreases as $1/r$.

(c) Looking at the electric fields, the first term is the static electric field of a dipole (as we derived in electrostatics), the last term is the radiation field of the static dipole.

(d) Looking at the magnetic field. The first term is what we derived in a quasi-static approximation, and the second term is the radiation field.

(e) The electric and magnetic duality says that the fields of a magnetic dipole can be found with the replacements $\boldsymbol{E} \to \boldsymbol{B}$, and $\boldsymbol{B} \to -\boldsymbol{E}$, $\boldsymbol{p} \to \boldsymbol{m}$

11.4 Attenas

(a) In an antenna with sinusoidal frequency we have

$$\boldsymbol{J}(T, \boldsymbol{r}_o) = e^{-i\omega(t - \frac{r}{c} + \frac{\boldsymbol{n} \cdot \boldsymbol{r}_o}{c})} \boldsymbol{J}(\boldsymbol{r}_o) \tag{11.59}$$

(b) Then the radiation field for a sinusoidal current is:

$$\boldsymbol{A}_{\text{rad}} = \frac{e^{-i\omega(t-r/c)}}{4\pi r} \int_{\boldsymbol{r}_o} e^{-i\omega \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}} \boldsymbol{J}(\boldsymbol{r}_o)/c \tag{11.60}$$

In general one will need to do this integral to determine the radiation field.

(c) The typical radiation resistance associated with driving a current which will radiate over a wide range of frequencies is $R_{\text{vacuum}} = c\mu_o = \sqrt{\mu_o/\epsilon_o} = 376\,\text{Ohm}$.

12 Relativity

Postulates

(a) All inertial observers have the same equations of motion and the same physical laws. Relativity explains how to translate the measurements and events according to one inertial observer to another.

(b) The speed of light is constant for all inertial frames

12.1 Elementary Relativity

Mechanics of indices, four-vectors, Lorentz transformations

(a) We desribe physics as a sequence of events labelled by their space time coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (c\,t, \boldsymbol{x}) \tag{12.1}$$

The space time coordinates of another inertial observer moving with velocity $\boldsymbol{v}$ relative to the first measures the coordinates of an event to be

$$\underline{x}^\mu = (\underline{x}^0, \underline{x}^1, \underline{x}^2 \underline{x}^3) = (\underline{c\,t}, \underline{\boldsymbol{x}}) \tag{12.2}$$

(b) The coordinates of an event according to the first observer x^μ determine the coordinates of an event according to another observer $\underline{x}^\mu$ through a linear change of coordinates known as a Lorentz transformation:

$$x^\mu \to \underline{x}^\mu = L^\mu_{\ \nu}(\boldsymbol{v}) x^\nu \tag{12.3}$$

I usually think of x^μ as a column vector

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{12.4}$$

so that without indices the transform

$$\big(x\big) \to \big(\underline{x}\big) = \big(\mathcal{L}\big)\ \big(x\big) \tag{12.5}$$

where $\mathcal{L}$ is the a matrix and (x) signifies column vectors like Eq. (12.4)

Then to change frames from K to an observer $\underline{K}$ moving to the right with speed v relative to K the transformation matrix is

$$(\mathcal{L}) = (L^\mu_{\ \nu}) = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad (\mathcal{L})^\mu_{\ \nu} = L^\mu_{\ \nu} \tag{12.6}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Here $L^0_{\ 1} = -\gamma\beta$ is the entry in the "0"-th row and "1"-st column

A short excercise done in class shows that a this boost contracts the $x^+ \equiv x^0 + x^1$ direction (*i.e.* $ct + x$) and expands the $x^- \equiv x^0 - x^1$ direction (*i.e.* $ct - x$). Thus, x^+ and x^- are eigenvectors of Lorentz boosts in the x direction

$$\underline{x}^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x^+ \tag{12.7}$$

$$\underline{x}^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^- \tag{12.8}$$

(c) Instead of using v we sometimes use the rapidity y

$$\tanh y = \frac{v}{c} \qquad \text{or} \qquad y = \tfrac{1}{2} \ln \frac{1+\beta}{1-\beta} \tag{12.9}$$

and note that $y \simeq \beta$ for small β

With this parametrization we find that the Lorentz boost appears as a hyperbolic rotation matrix

$$(\mathcal{L}) = (L^{\mu}{}_{\nu}) = \begin{pmatrix} \cosh y & -\sinh y & & \\ -\sinh y & \cosh y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{12.10}$$

Then

$$\underline{x}^+ = e^{-y} x^+ \qquad \underline{x}^- = e^{y} x^- \tag{12.11}$$

(d) Since the spead of light is constant for all observers we demand that

$$-(ct)^2 + \boldsymbol{x}^2 = -\underline{(ct)}^2 + \underline{\boldsymbol{x}}^2 \tag{12.12}$$

under Lorentz transformation. We also require that the set of Lorentz transformations satisfy the follow (group) requirements:

$$\mathcal{L}(-\boldsymbol{v})\mathcal{L}(\boldsymbol{v}) = \mathbb{I} \tag{12.13}$$

$$\mathcal{L}(\boldsymbol{v}_2)\mathcal{L}(\boldsymbol{v}_1) = \mathcal{L}(\boldsymbol{v}_3) \tag{12.14}$$

here $\mathbb{I}$ is the identity matrix. These properties seem reasonable to me, since if I transform to frame moving with velocity $\boldsymbol{v}$ and then transform back to a frame moving with veloicty $-\boldsymbol{v}$, I shuld get back the same result. Similarly two Lorentz transformations produce another Lorentz transformation.

(e) Since the combination

$$-(ct)^2 + \boldsymbol{x}^2 \tag{12.15}$$

is invariant under lorentz transformation, we introduced an index notation to make such invariant forms manifest. We formalized the lowering of indices

$$x_\mu = g_{\mu\nu} x^\nu \qquad x_\mu = (-c\,t, \boldsymbol{x}) \tag{12.16}$$

with a metric tensor:

$$g_{00} = -1 \qquad g_{11} = g_{22} = g_{33} = 1 \tag{12.17}$$

In this way we define a dot product

$$x \cdot x = x^\mu x_\mu = -(ct)^2 + \boldsymbol{x}^2 \tag{12.18}$$

is manifestly invariant.

Similarly we raise indices

$$x^\mu = g^{\mu\nu} x_\nu \tag{12.19}$$

with

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{12.20}$$

Of course the process of lowering and index and then raising it agiain does nothing:

$$g^{\mu}_{\nu} = g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu} = \text{identity matrix } = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{12.21}$$

(f) Generally the upper indices are "the normal thing". We will try to leave the dimensions and name of the four vector, corresponding to that of the spatial components. Examples: $x^\mu = (ct, \boldsymbol{x})$, $A^\mu = (\Phi, \boldsymbol{A})$, $J^\mu = (c\rho, \boldsymbol{j})$, and $P^\mu = (E/c, \boldsymbol{p})$.

(g) Four vectors are anything that transforms according to the lorentz transformation $A^\mu = (A^0, \boldsymbol{A})$ like coordinates

$$A^\mu = L^\mu_\nu A^\nu \tag{12.22}$$

Given two four vectors, A^μ and B^μ one can always construct a Lorentz invariant quantity.

$$A \cdot B = A_\mu B^\mu = A^\mu g_{\mu\nu} B^\nu = -A^0 B^0 + \boldsymbol{A}\cdot\boldsymbol{B} = -\underline{A}^0 \underline{B}^0 + \underline{\boldsymbol{A}} \cdot \underline{\boldsymbol{B}} = \underline{A}^\mu g_{\mu\nu} \underline{B}^\mu = \underline{A}_\mu \underline{B}^\mu = \underline{A} \cdot \underline{B} \tag{12.23}$$

(h) Notation. We denote the transformation *matrix*

$$(\mathcal{L}) \tag{12.24}$$

A matrix just has rows and columns and has no idea what is a row with an upper index μ versus a lower index

Then *entries* $(\mathcal{L})_{\mu\nu}$ of the matrix are labelled by rows (μ) and columns (ν). You are free to move this row and column index up and down at will – the first index labels the row, the second the column. In this way

$$(\mathcal{L})_{\mu\nu} = \left(\mathcal{L}^\top\right)_{\nu\mu} = (\mathcal{L})^{\mu}{}_{\nu} = \left(\mathcal{L}^\top\right)_{\nu}{}^{\mu} = L^{\mu}{}_{\nu} \tag{12.25}$$

is all the same numerical number $L^\mu{}_\nu$ for specified μ and ν. However, the much *preferred* placemnt of the indices surrounding the matrix is just a visual reminder of the individual entries L^μ_ν which together form the matrix, $(\mathcal{L})$ and $(\mathcal{L}^\top)$, and that is all, *e.g.*

$$\underline{x}^\mu = L^\mu{}_\nu x^\nu = (\mathcal{L})^\mu{}_\nu x^\nu = x^\nu (\mathcal{L}^\top)_\nu{}^\mu \tag{12.26}$$

The indices labelling $L^\mu{}_\nu$ can not be raised and lowered randomly, but are raised and lowered with the metric tensor, i.e. multiplying the matrix $(\mathcal{L})$ with the matrix (g). Thus

$$(g\mathcal{L})_{\mu\nu} = g_{\mu\rho} L^\rho{}_\nu \equiv L_{\mu\nu} \tag{12.27}$$

and

$$(g\mathcal{L}g)_\mu{}^\nu = g_{\mu\rho} L^\rho{}_\sigma g^{\sigma\nu} \equiv L_\mu{}^\nu \tag{12.28}$$

(i) From the invariance of the inner prodcut we see that the lower (covariant) components of four vectors transform with the inverse transformation and as a row,

$$x_\mu \to \underline{x}_\nu = x_\mu (\mathcal{L}^{-1})^\mu_\nu\,. \tag{12.29}$$

I usually think of x_μ (with a lower index) as a row

$$(x_0\ x_1\ x_2\ x_3) \tag{12.30}$$

So the transformation rule in terms of matrices is

$$(\underline{x}_0\ \underline{x}_1\ \underline{x}_2\ \underline{x}_3) = (x_0\ x_1\ x_2\ x_3)\begin{pmatrix} \mathcal{L}^{-1} \end{pmatrix} \tag{12.31}$$

In this way the inner product

$$\underline{A}_\mu \underline{B}^\mu = (A_0\ A_1\ A_2\ A_3)\begin{pmatrix} \mathcal{L}^{-1} \end{pmatrix}\begin{pmatrix} \mathcal{L} \end{pmatrix}\begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} = A_\mu B^\mu \tag{12.32}$$

is invariant. If you wish to think of x_μ as a column, then it transforms under lorentz transformation with the inverse transpose matrix

$$\begin{pmatrix} \underline{x}_0 \\ \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{pmatrix} = \begin{pmatrix} \mathcal{L}^{-1\top} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{12.33}$$

(j) As is clear from Eq. (12.23), the metric tensor is an invariant tensor, *i.e.*

$$g^{\mu\nu} = L^\mu_{\ \rho} L^\nu_{\ \sigma} g^{\rho\sigma} = (\mathcal{L})^\mu_{\ \rho}(\mathcal{L})^\nu_{\ \sigma} g^{\rho\sigma} \tag{12.34}$$

is the same tensor diag$(-1, 1, 1, 1)$ in all frames (so I dont need to put an underline $\underline{g}^{\mu\nu}$ on the LHS). From Eq. (12.34) it follows that the inverse (transpose) Lorentz transform can be found by raising and lowering the indices of the transform matrix, *i.e.*

$$L_\rho^{\ \sigma} \equiv g_{\rho\mu} L^\mu_{\ \nu} g^{\nu\sigma} = (\mathcal{L}^{-1\top})_\rho^{\ \sigma} = (\mathcal{L}^{-1})^\sigma_{\ \rho} \tag{12.35}$$

where we have defined $L_\rho^{\ \sigma}$. Thus if one wishes to think of a lowered four vector A_μ as a column, one has

$$\underline{A}_\nu = L_\nu^{\ \mu} A_\mu \tag{12.36}$$

Thus, a short excercise (done) in class shows that if

$$\underline{T}^{\mu\nu} = L^\nu_{\ \rho} L^\mu_{\ \sigma} T^{\sigma\rho} \tag{12.37}$$

$$= (\mathcal{L})^\mu_{\ \sigma}\, T^{\sigma\rho}\, (\mathcal{L}^\top)_\rho^{\ \nu} \tag{12.38}$$

then there is a consistency check

$$\underline{T}^\mu_{\ \nu} = L^\mu_{\ \sigma}\, L_\nu^{\ \rho}\, T^\sigma_{\ \rho} \tag{12.39}$$

$$= (\mathcal{L})^\mu_{\ \sigma}\, T^\sigma_{\ \rho}\, (\mathcal{L}^{-1})^\rho_{\ \nu} \tag{12.40}$$

i.e. that lower indices transform like rows with the inverse matrix ($\mathcal{L}^{-1}$) upstairs indices transform like columns with the regular matrix ($\mathcal{L}$).

Doppler shift, four velocity, and proper time.

(a) The frequency and wave number form a four vector $K^\mu = (\frac{\omega}{c}, \boldsymbol{k})$, with $|\boldsymbol{k}| = \omega/c$. This can be used to determine a relativistic dopler shift.

(b) For a particle in motion with velocity $v_{\boldsymbol{p}}$ and gamma factor $\gamma_{\boldsymbol{p}}$, the space-time interval is

$$ds^2 \equiv dx_\mu dx^\mu = -(cdt)^2 + d\boldsymbol{x}^2 = -(cd\tau)^2\,. \tag{12.41}$$

ds^2 is associated with the clicks of the clock in the particles instantaneous rest frame, $ds^2 = -(cd\tau)^2$, so we have in any other frame

$$d\tau \equiv \sqrt{-ds^2}/c = dt\sqrt{1 - \left(\frac{dx}{dt}\right)^2/c^2} \tag{12.42}$$

$$= \frac{dt}{\gamma_{\boldsymbol{p}}} \tag{12.43}$$

(c) The four velocity of a particle is the distance the particle travels per proper time

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = (u^0, \boldsymbol{u}) = (\gamma_p c, \gamma_p \boldsymbol{v}_p) \tag{12.44}$$

so

$$\underline{U}^\mu = L^\mu_{\ \nu} U^\nu \tag{12.45}$$

Note $U_\mu U^\mu = -c^2$.

(d) The transformation of the four velocity under Lorentz transformation should be compared to the transformation of velocities. For a particle moving with velocity $\boldsymbol{v}_p$ in frame K, then in another frame $\underline{K}$ moving to the right with speed v the particle moves with velocity

$$\underline{v}_p^{\parallel} = \frac{v_p^{\parallel} - v}{1 - v_p^{\parallel} v/c^2} \tag{12.46}$$

$$\underline{v}_p^{\perp} = \frac{v_p^{\perp}}{\gamma_p(1 - v_p^{\parallel} v/c^2)} \tag{12.47}$$

where $v_p^{\parallel}$ and $v_p^{\perp}$ are the components of $\boldsymbol{v}_p$ parallel and perpendicular to v. These are easily derived from the transformation rules of U^μ and the fact that $\boldsymbol{v}_p = \boldsymbol{u}/u^0$.

Energy and Momentum Conservation

(a) Finally the energy and momentum form a four vector

$$P^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) \tag{12.48}$$

The invariant product of P^μ with itsself the rest energy

$$P^\mu P_\mu = -(mc)^2 \tag{12.49}$$

This can be inverted giving the energy in terms of the momentum, *i.e.* the dispersion curve

$$\frac{E(p)}{c} = \sqrt{p^2 + (mc)^2} \tag{12.50}$$

(b) The relation between energy and momentum determines the velocity. At rest $E = mc^2$. Then a boost in the negative $-\boldsymbol{v}_p$ direction shows that a particle with velocity $\boldsymbol{v}_p$ has energy and momentum

$$P^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) = mc\,(\gamma_p, \gamma_p \beta_{\boldsymbol{p}}) = mU^\mu \tag{12.51}$$

i.e.

$$v_{\boldsymbol{p}} = c\,\frac{p}{(E/c)} = \frac{\partial E(p)}{\partial p} \tag{12.52}$$

Thus as usual the derivative of the dispersion curve is the velocity.

(c) Energy and Momentum are conserved in collisions, e.g. for a reaction $1 + 2 \to 3 + 4$ w have

$$P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu \tag{12.53}$$

Usually when working with collisions it makes sense to suppress c or just make the association:

$$\begin{pmatrix} E \\ p \\ m \end{pmatrix} \qquad \text{is short for} \qquad \begin{pmatrix} E \\ cp \\ mc^2 \end{pmatrix} \tag{12.54}$$

A starting point for analyzing the kinematics of a process is to "square" both sides with the invariant dot product $P^2 \equiv P \cdot P$. For example if $P_1 + P_2 = P_3 + P_4$ then:

$$(P_1 + P_2)^2 = (P_3 + P_4)^2 \tag{12.55}$$

$$P_1^2 + P_2^2 + 2P_1 \cdot P_2 = P_3^2 + P_4^2 + 2P_3 \cdot P_4 \tag{12.56}$$

$$-m_1^2 - m_2^2 - 2E_1E_2 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = -m_3^2 - m_4^2 - 2E_3E_4 + 2\boldsymbol{p}_3 \cdot \boldsymbol{p}_4 \tag{12.57}$$

12.2 Covariant form of electrodynamics

(a) The players are:

i) The derivatives

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{12.58}$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{12.59}$$

ii) The wave operator

$$\Box = \partial_\mu \partial^\mu = \frac{-1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 \tag{12.60}$$

iii) The four velocity $U^\mu = (u^0, \boldsymbol{u}) = (\gamma_p, \gamma_p \boldsymbol{v}_p)$

iv) The current four vector

$$J^\mu = (c\rho, \boldsymbol{J}) \tag{12.61}$$

v) The vector potential

$$A^\mu = (\Phi, \boldsymbol{A}) \tag{12.62}$$

vi) The field strength is a tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{12.63}$$

which ultimately comes from the relations

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{12.64}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{12.65}$$

In indices we have

$$F^{0i} = E^i \qquad\qquad E^i = F^{0i} \tag{12.66}$$

$$F^{ij} = \epsilon^{ijk} B_k \qquad\qquad B_i = \tfrac{1}{2}\epsilon_{ijk} F^{jk} \tag{12.67}$$

In matrix form this anti-symmetric tensor reads

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \tag{12.68}$$

Raising and lowering indices of $F^{\mu\nu}$ can change the sign of the zero components, but does not change the ij components, *e.g.*

$$E^i = F^{0i} = -F^{i0} = F^i{}_0 = -F_0{}^i = -F_{0i} = F^0{}_i = F^{0i} \tag{12.69}$$

vii) The dual field tensor implements the replacement

$$\boldsymbol{E} \to \boldsymbol{B} \qquad \boldsymbol{B} \to -\boldsymbol{E} \tag{12.70}$$

As motivated by the maxwell equations in free space

$$\nabla \cdot \boldsymbol{E} = 0 \tag{12.71}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = 0 \tag{12.72}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{12.73}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0 \tag{12.74}$$

which are the same before and after this duality transformation. The dual field stength tensor is

$$\mathscr{F}^{\alpha\beta} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & -E^x & 0 \end{pmatrix} \tag{12.75}$$

The dual field strength tensor

$$\mathscr{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{12.76}$$

where the totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even perms 0,1,2,3} \\ -1 & \text{odd perms 0,1,2,3} \\ 0 & \text{0 otherwise} \end{cases} \tag{12.77}$$

viii) The stress tensor is

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\lambda}F^{\nu}_{\ \lambda} + g^{\mu\nu}\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{12.78}$$

Or in terms of matrices

$$\Theta^{\mu\nu}_{\rm em} = \left(\begin{array}{c|c} u_{\rm em} & \boldsymbol{S}_{\rm em}/c \\ \hline \boldsymbol{S}_{\rm em}/c & T^{ij} \end{array}\right) \tag{12.79}$$

Note that $\Theta^{0i} = \boldsymbol{S}^i_{\rm em}/c = c\,\boldsymbol{g}^i_{\rm em}$, and $T^{ij} = (-E^iE^j + \frac{1}{2}\delta^{ij}E^2) + (-B^iB^j + \frac{1}{2}\delta^{ij}B^2)$. You can remember the stress tensor $\Theta^{\mu\nu}$ by recalling that it is quadratic in F, symmetric under interchange of μ and ν, and traceless $\Theta^{\mu}_{\ \mu} = 0$. These properties fix the stress tensor up to a constant.

(b) The equations are

i) The continuity equation:

$$\partial_\mu J^\mu = 0 \tag{12.80}$$

$$\partial_t \rho + \nabla\cdot\boldsymbol{J} = 0 \tag{12.81}$$

ii) The wave equation in the covariant gauge

$$-\Box A^\mu = J^\mu/c \tag{12.82}$$

$$-\Box\Phi = \rho \tag{12.83}$$

$$-\Box\boldsymbol{A} = \boldsymbol{J}/c \tag{12.84}$$

This is true in the covariant gauge

$$\partial_\mu A^\mu = 0 \tag{12.85}$$

$$\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A} = 0 \tag{12.86}$$

iii) The force law is:

$$\frac{dP^\mu}{d\tau} = eF^{\mu}_{\ \nu}\frac{U^\nu}{c} \tag{12.87}$$

$$\frac{1}{c}\frac{dE}{dt} = e\boldsymbol{E}\cdot\frac{\boldsymbol{v}}{c} \tag{12.88}$$

$$\frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E} + e\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} \tag{12.89}$$

If these equations are multiplied by γ they equal the relativistic equations to the left.

iv) The sourced field equations are :

$$-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c} \tag{12.90}$$

$$\nabla \cdot \boldsymbol{E} = \rho \tag{12.91}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} \tag{12.92}$$

v) The dual field equations are :

$$-\partial_\mu \mathscr{F}^{\mu\nu} = 0 \tag{12.93}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{12.94}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0 \tag{12.95}$$

as might have been inferred by the replacements $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$. The dual field equations can also be written in terms $F_{\mu\nu}$, and this is known as the Bianchi identity:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0\,, \tag{12.96}$$

where ρ, μ, ν are cyclic.
Or (for the mathematically inclined) the Bianchi identity reads

$$\partial_{[\mu_1} F_{\mu_2\mu_3]} = 0\,, \tag{12.97}$$

where the square brackets denote the fully antisymmetric combination of μ_1, μ_2, μ_2, *i.e.* the order is like a determinant

$$\partial_{[\mu_1} F_{\mu_2\mu_3]} \equiv \frac{1}{3!}\big[(\partial_{\mu_1} F_{\mu_2\mu_3} - \partial_{\mu_2} F_{\mu_1\mu_3} + \partial_{\mu_3} F_{\mu_1\mu_2}) + (-\partial_{\mu_1} F_{\mu_3\mu_2} + \partial_{\mu_2} F_{\mu_3\mu_1} - \partial_{\mu_3} F_{\mu_2\mu_1})\big] \tag{12.98}$$

The second line is the same as the first since $F_{\mu\nu}$ is antisymmetric. Eq. (12.97) is the statement that $F_{\mu\nu}$ is an exact differential form.

vi) The dual field equations are equivalent to the statement that that $F_{\mu\nu}$ (or $\boldsymbol{E}, \boldsymbol{B}$) can be written in terms of the gauge potential A_μ (or $\Phi, \boldsymbol{A}$)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{12.99}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{12.100}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{12.101}$$

The potentials are not unique as we can always make a gauge transform:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{12.102}$$

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\Lambda \tag{12.103}$$

$$\Phi \to \Phi + \frac{1}{c}\partial_t \Lambda \tag{12.104}$$

vii) The conservation of energy and momentum can be written in terms of the stress tensor:

$$-\partial_\mu \Theta^{\mu\nu}_{\rm em} = F^{\mu}_{\ \nu}\frac{J^\nu}{c} \tag{12.105}$$

$$-\left(\frac{1}{c}\frac{\partial u_{\rm em}}{\partial t} + \nabla \cdot (\boldsymbol{S}_{\rm em}/c)\right) = \boldsymbol{E} \cdot \boldsymbol{J}/c \tag{12.106}$$

$$-\left(\frac{1}{c}\frac{\partial \boldsymbol{S}^j_{\rm em}/c}{\partial t} + \partial_i T^{ij}\right) = \rho E^j + (\boldsymbol{J}/c \times \boldsymbol{B})^j \tag{12.107}$$

The energy and momentum transferred from the fields $F^{\mu\nu}$ to the particles is

$$\partial_\mu \Theta^{\mu\nu}_{\rm mech} = F^{\mu}_{\ \nu}\frac{J^\nu}{c} \tag{12.108}$$

Or

$$\partial_\mu \Theta^{\mu\nu}_{\text{mech}} + \partial_\mu \Theta^{\mu\nu}_{\text{em}} = 0 \tag{12.109}$$

12.3 Transformation of field strengths

(a) By using the lorentz transformation rule

$$\underline{F}^{\mu\nu} = L^{\mu}_{\ \rho} L^{\nu}_{\ \sigma} F^{\rho\sigma} \tag{12.110}$$

We deduced the transformation rule for the change of $F^{\rho\sigma}$ under a change of frame (boost). The $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ fields in frame $\underline{K}$, which is moving with velocity $\boldsymbol{v}/c = \boldsymbol{\beta}$ relative to a frame K, are related to the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in frame K via

$$\underline{E}_{\parallel} = E_{\parallel} \qquad \underline{B}_{\parallel} = B_{\parallel} \tag{12.111}$$

$$\underline{\boldsymbol{E}}_{\perp} = \gamma \boldsymbol{E}_{\perp} + \gamma \boldsymbol{\beta} \times \boldsymbol{B}_{\perp} \qquad \underline{\boldsymbol{B}}_{\perp} = \gamma \boldsymbol{B}_{\perp} - \gamma \boldsymbol{\beta} \times \boldsymbol{E}_{\perp} \tag{12.112}$$

where $E_{\parallel}$ and $B_{\parallel}$ are the components of the E and B fields parallel to the boost, while $\boldsymbol{E}_{\perp}$ and $\boldsymbol{B}_{\perp}$ are the components of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields perpendicular to the boost.

(b) The quadratic invariants of $F_{\mu\nu}$ are

$$F_{\mu\nu} F^{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) \tag{12.113}$$

$$F_{\mu\nu} \mathscr{F}^{\mu\nu} = -4\boldsymbol{E} \cdot \boldsymbol{B} \tag{12.114}$$

Thus, if the electric and magnetic fields are orthogonal in one frame, then they are orthogonal in all. In particular, if the field is electrostatic in one a particular frame ($\boldsymbol{B} = 0$), then $F_{\mu\nu}F^{\mu\nu}$ is negative in all frames, and $\boldsymbol{E}$ will be perpendicular to $\boldsymbol{B}$ in all frames.

(c) If in the lab frame there is only an electric field $\boldsymbol{E}$, then the transformation rule of $F_{\mu\nu}$ is often used to determine the magnetic field which is experienced by a slow moving charge of velocity $\boldsymbol{v}/c = \boldsymbol{\beta}$

$$\boldsymbol{B} = -\boldsymbol{\beta} \times \boldsymbol{E} \tag{12.115}$$

(d) We used the transformation rule to determine the (boosted) Coulomb fields for a fast moving charge. For a charge moving along the x-axis crossing the origin $x = 0$ at time $t = 0$, the fields at longitidunal coordinate x and transverse coordinates $\boldsymbol{b} = (y, z)$ we found

$$E_{\parallel}(t, x, \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma(x - v_p t)}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{12.116}$$

$$\boldsymbol{E}_{\perp}(t, x, \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma \boldsymbol{b}}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{12.117}$$

$$\boldsymbol{B} = \frac{\boldsymbol{v_p}}{c} \times \boldsymbol{E} \tag{12.118}$$

Note that in Eqs. 12.111, $\boldsymbol{\beta}$ is the velocity of the frame $\underline{K}$ relative to K. In this case we know the fields of in the frame of the particle (the Coulomb field), and we want to know the fields in a frame $\underline{K}$ (the lab) moving with speed $\boldsymbol{\beta} = -\boldsymbol{v}_p$ relative to the particle. The frame $\underline{K}$ (the lab) sees the particle moving with velocity $\boldsymbol{v}_p$. Thus, we make a Lorentz transform as in Eq. (12.111) with $\boldsymbol{\beta} = -\boldsymbol{v}_p$ to transform from the particle frame to the lab frame.

(e) The constituent relation specifies the current $\boldsymbol{j}$ of the sample in terms of the applied fields. In particular, for a conductor we explained that $\boldsymbol{j} = \sigma \boldsymbol{E}$ in the rest frame of the conductor. Boosting this relationship, we found that for samples moving non-relativistically with speed $\boldsymbol{v}$ relative to the lab, that the constituent relation takes form

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}) \tag{12.119}$$

where $\boldsymbol{v}$ is the velocity of the sample.

12.4 Covariant actions and equations of motion

(a) We discussed the simplest of all actions

$$I[x(t)] = \underbrace{I_o}_{\text{free}} + \underbrace{I_{\text{int}}}_{\text{interaction}} \,, \tag{12.120}$$

$$= \underbrace{\int dt\, \tfrac{1}{2} m\dot{x}^2(t)}_{\text{free}} + \underbrace{\int dt\, F_o\, x(t)}_{\text{interaction}} \tag{12.121}$$

we varied this, and derived Newton's Law. All other actions follow this model.

(b) For a relativistic point particle interaction with the electromagnetic field we derived a Lorentz covariant free and interation lagrangian:

i) The free part of the action is

$$I_o = -\int d\tau\, mc^2 \tag{12.122}$$

Using

$$c\, d\tau = \sqrt{-dX^\mu dX_\mu} \tag{12.123}$$

we have

$$I_o[X^\mu(p)] = -\int d\tau\, mc^2 = \int dp\, mc\,\sqrt{-\frac{dX^\mu}{dp}\frac{dX_\mu}{dp}} \tag{12.124}$$

We derived the equations of motion by varying this action $X^\mu(p) \to X^\mu(p) + \delta X^\mu(p)$

ii) The interaction Lagrangian for a charged particle is

$$I_{\text{int}}[X^\mu(p)] = \frac{e}{c}\int dp\, \frac{dX^\mu}{dp} A_\mu(X(p)) \tag{12.125}$$

or in terms of proper time

$$I_{\text{int}}[X^\mu(\tau)] = \frac{e}{c}\int d\tau\, \frac{dX^\mu}{d\tau} A_\mu(X(\tau)) \tag{12.126}$$

A one line exercise shows that a gauge transformation (with $\Lambda(x)$ that vanishes as $x \to \pm\infty$), leaves the action unchanged.
In the non-relativistic limit this reduces to

$$I_{\text{int}}[\boldsymbol{x}(t)] = \int dt\, \left[-e\Phi(t, \boldsymbol{x}(t)) + \frac{\boldsymbol{v}}{c}\cdot \boldsymbol{A}(t, \boldsymbol{x}(t))\right] \tag{12.127}$$

iii) Varying the free and interaction actions with respect to $X^\mu \to X^\mu + \delta X^\mu$

$$\delta I[X] = \delta I_o + \delta I_{\text{int}} \tag{12.128}$$

we found the equations of motion

$$m\frac{d^2X^\mu}{d\tau^2} = eF^\mu_{\ \nu}\frac{U^\nu}{c} \tag{12.129}$$

(c) We also wrote down the action for the fields

i) The unique action, which is invariant under Lorentz transformations, gauge gauge transformations, and parity, that involves no more than two powers of the field strength is

$$I_o = \int d^4x\, \frac{-1}{4} F_{\mu\nu}F^{\mu\nu} \tag{12.130}$$

ii) The interaction between the currents and the fields is

$$I_{\rm int} = \int d^4x \, J^\mu \frac{A_\mu}{c} \tag{12.131}$$

Indeed, for any particular gauge invariant interaction Lagrangian (such as Eq. (12.126)) the (current)/c is defined to be the variation of the interaction Lagrangian with respect to A_μ

$$\delta I_{\rm int} = \int d^4x \underbrace{\frac{J^\mu(x)}{c}}_{\text{definition of current}/c} \delta A_\mu(x) \tag{12.132}$$

For the point particle action Eq. (12.126), this gives

$$\frac{J^\mu}{c} = e(\delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t)), \boldsymbol{\beta}\delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t))) \tag{12.133}$$

where $\boldsymbol{x}_o(t)$ is the position of the particle.

iii) Varying the complete action

$$\delta I_{\rm tot} = \delta I_o + \delta I_{\rm int} \tag{12.134}$$

Yields the Maxwell equations

$$-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c} \tag{12.135}$$

iv) Demanding that the interaction part of the action $I_{\rm int}$ is invariant under gauge transformation leads to a requirement of current conservation:

$$\partial_\mu J^\mu = 0 \tag{12.136}$$

Similarly if $\partial_\mu J^\mu = 0$, then a gauge transformation leaves Eq. (12.131) unchanged.

13 Radiation from Relativistic Charged Particles

13.1 Basic equations

(a) We wrote down the wave equations in the covariant gauge:

$$-\Box\Phi = \rho(t_o, \boldsymbol{r}_o) \tag{13.1}$$

$$-\Box\boldsymbol{A} = \boldsymbol{J}(t_o, \boldsymbol{r}_o)/c \tag{13.2}$$

(b) Then we used the green function of the wave equation

$$G(t, r|t_o r_o) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t - t_o + \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c}) \tag{13.3}$$

to determine the potentials $(\Phi, \boldsymbol{A})$ with the current

$$\frac{J^\mu}{c} = (\rho, \frac{\boldsymbol{J}}{c}) = (q\,\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_o))\ ,\ q\,\frac{\boldsymbol{v}(t_o)}{c}\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_o))) \tag{13.4}$$

This yields the Lienard-Wiechert potentials

$$\Phi = \frac{q}{4\pi|\boldsymbol{r}-\boldsymbol{r}_*(T)|}\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \Longrightarrow \frac{q}{4\pi r}\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \tag{13.5}$$

$$\boldsymbol{A} = \frac{q}{4\pi|\boldsymbol{r}-\boldsymbol{r}_*(T)|}\frac{\boldsymbol{\beta}(T)}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \Longrightarrow \frac{q}{4\pi r}\frac{\boldsymbol{\beta}(T)}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \tag{13.6}$$

where the retarded time is

$$T(t, r) = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_*(T)|}{c} \Longrightarrow T(t, r) = t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_*(T)}{c} \tag{13.7}$$

The terms after the Longrightarrow indicate the far field limit

(c) The Lienard Wiechert potential can also be obtained by integrating over $\boldsymbol{r}_o$ in Eq. (11.8).

(d) The factor "collinear facor" (my name), or dT/dt

$$\frac{dT}{dt} = \frac{1}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})} \tag{13.8}$$

$$\frac{dT}{d\boldsymbol{r}^i} = \frac{1}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}\frac{-\boldsymbol{n}_i}{c} \tag{13.9}$$

is quite important. We gave a physical interpretation of it in class. If a wave form is *observed* to have a time scale of Δt, then the *formation time* of the wave, ΔT, is

$$\Delta T = \frac{dT}{dt}\Delta t = \frac{\Delta t}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}} \tag{13.10}$$

In particular, a fourier component with frequency ω in the observed wave was formed over the time

$$\Delta T \sim \frac{1}{\omega(1-\boldsymbol{n}\cdot\boldsymbol{\beta})} \tag{13.11}$$

(e) In the ultrarelativistic limit $1/(1-\beta\cos\theta)$ is often approximated for $\theta \ll 1$ and for ultra-relativistic particles $1-\beta \simeq 1/2\gamma^2$

$$\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}} = \frac{2\gamma^2}{1+(\gamma\theta)^2} \tag{13.12}$$

(f) The magnetic and electric fields can be determined from $\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A}_{\rm rad} - \nabla\Phi$. As discussed in a separate note ("retarded_time.pdf"), In the far field limit this is the same as computing

$$\boldsymbol{E}(t,r) = \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{c}\partial_t \boldsymbol{A}_{\rm rad}(T) \tag{13.13a}$$

$$= \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\,\frac{1}{c}\frac{\partial}{\partial T}\boldsymbol{A}_{\rm rad}(T) \tag{13.13b}$$

$$= \frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\,\frac{1}{c}\frac{\partial}{\partial T}\left[\frac{q}{4\pi r}\,\frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\right]_{\rm ret} \tag{13.13c}$$

$$= \frac{q}{4\pi r c^2}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} \tag{13.13d}$$

The $[]_{\rm ret}$ indicates that the velocity and acceleration are to be evaluated at the retarded time $T(t,r)$.

The magnetic field is

$$\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E} \tag{13.14}$$

For below, it is worth noting below that

$$\frac{1}{c}\frac{\partial}{\partial T}\left[\boldsymbol{n}\times\boldsymbol{n}\times A_{\rm rad}\right] = \frac{1}{c}\frac{\partial}{\partial T}\left[\frac{q}{4\pi r}\,\frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}\right] \tag{13.15}$$

$$= \frac{q}{4\pi r c^2}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}\right]_{\rm ret} \tag{13.16}$$

(g) We will often be interested in the frequency distribution of the radiation.

$$\boldsymbol{E}(\omega,r) \equiv \int_{-\infty}^{\infty} dt\, e^{i\omega t}\boldsymbol{E}(t,r) \tag{13.17a}$$

$$= \frac{q\, e^{i\omega r/c}}{4\pi r c^2}\int_{-\infty}^{\infty} dT\, e^{i\omega(T-\boldsymbol{n}\cdot\boldsymbol{r}_*(T)/c)}\,\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2} \tag{13.17b}$$

$$= \frac{q\,(-i\omega\, e^{i\omega r/c})}{4\pi r c}\int_{-\infty}^{\infty} dT\, e^{i\omega(T-\boldsymbol{n}\cdot\boldsymbol{r}_*(T)/c)}\;\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta} \tag{13.17c}$$

We are computing the fourier transfrom of $\boldsymbol{E}_{\rm rad}(t,\boldsymbol{r})$ to find $\boldsymbol{E}_{\rm rad}(\omega,r)$. Changing variables to integrate over T instead of t yields Eq. (13.17b) with Eq. (13.13d). Integrating by parts using Eq. (13.15) yields Eq. (13.17c). This final form Eq. (13.17c) is often the most convenient, but sometimes it is just easier to use Eq. (13.17b) which shows explicity the dependence on acceleration.

Observables in the far field

(a) The energy per time per solid angle received at the detector is

$$\frac{dW}{dt d\Omega} = \frac{dP(t)}{d\Omega} = r^2\boldsymbol{S}\cdot n \tag{13.18}$$

$$= c|r\boldsymbol{E}|^2 \tag{13.19}$$

This is what you want to know if you want to find out if the detector will burn up.

(b) We often wan't to know how much energy was radiated over a given period of acceleration, $T_1 \dots T_2$. For example how much energy was lost by the particle as it moved through one complete circle. Then we want to evaluate the energy radiated per retarded time from T_1 up to the time it completes the circle T_2

$$\frac{dW}{dTd\Omega} = \frac{dP(T)}{d\Omega} = r^2 \boldsymbol{S} \cdot \boldsymbol{n} \frac{dt}{dT} \tag{13.20}$$

$$= c|rE|^2(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}) \tag{13.21}$$

(c) We are also interested in the frequency distribution of the emitted radiation. The energy per $d\omega/(2\pi)$ per solid angle is

$$(2\pi)\frac{dW}{d\omega d\Omega} \equiv c|rE(\omega, r)|^2 \tag{13.22}$$

Since the sign of the ω is without significance (for real fields such as the electromagnetic fields), we sometimes use

$$\frac{dI}{d\omega d\Omega} \equiv \frac{c|rE(\omega, r)|^2}{2\pi} + \frac{c|rE(-\omega, r)|^2}{2\pi} = \frac{c|rE(\omega, r)|^2}{\pi} \tag{13.23}$$

So that

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{dI}{d\omega d\Omega} \tag{13.24}$$

(d) The energy spectrum can be interperted as the average number of photons per frequency per solid angle

$$\frac{dI}{d\omega d\Omega} = \hbar\omega \frac{d\overline{N}}{d\omega d\Omega} \tag{13.25}$$

13.2 Relativistic Larmour

(a) For a particle undergoing arbitrary relativistic motion, we evaluated the energy per retarded time per solid angle

$$\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{|\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}|^2}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^5} \tag{13.26}$$

(b) Integrating over angles we get

$$P(T) = \frac{dW}{dT} = \frac{q^2}{4\pi} \frac{2}{3c^3} \gamma^6 \left[a_\parallel^2 + \frac{a_\perp^2}{\gamma^2} \right] \tag{13.27}$$

where $a_\parallel$ is the projection of $\boldsymbol{a} = d^2\boldsymbol{x}/dt^2$ along the direction of motion, and $a_\perp$ is the component of $\boldsymbol{a}$ perpendicular to the direction of motion, *i.e.* for $\boldsymbol{v}$ in the z direction

$$\boldsymbol{a} = (a_\perp^x, a_\perp^y, a_\parallel) \tag{13.28}$$

(c) The acceleration four vector is

$$\mathscr{A}^\mu = \frac{d^2 x^\mu}{d\tau^2} \tag{13.29}$$

For a paraticle moving along in the z-direction, the acceleration in the particle's locally inertial frame (i.e. the frame that is instantaneously moving with the particle) is

$$(\mathscr{A}^0, \mathscr{A}^1, \mathscr{A}^2, \mathscr{A}^3)\big|_{\text{rest frame}} = (0, \alpha_\perp^x, \alpha_\perp^y, \alpha_\parallel) \tag{13.30}$$

While in the lab frame $\mathscr{A}^\mu$ is found by boosting this result. The acceleration $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ is found from this result and the definition of propper time $d\tau = dt/\gamma$,

$$\boldsymbol{a} = (a_\perp^x, a_\perp^y, a_\parallel) = (\gamma^2 \alpha_\perp^x,\ \gamma^2 \alpha_\perp^y,\ \gamma^3 \alpha_\parallel) \tag{13.31}$$

You should be able to prove this. The relativistic Larmour fourmula can then be written

$$P(T) = \frac{q^2}{4\pi} \frac{2}{3c^3} \mathscr{A}_\mu \mathscr{A}^\mu \tag{13.32}$$

(d) For straight line acceleration at very large γ, we found that that the radiation is emitted within a cone of order

$$\Delta\Theta \sim 1/\gamma\,. \tag{13.33}$$

For θ very small $\theta \sim 1/\gamma$ we found,

$$\frac{dP(T)}{d\Omega} = \frac{2q^2}{\pi^2}\frac{a^2}{c^3}\gamma^8\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}\,. \tag{13.34}$$

You should feel comfortable deriving this result.

13.3 Synchrotron Radiation

(a) For a relativistic particle moving in a circle. The particle emmits light beamed in its direction of motion. Thus, an observer a large distance aweay from the rotational source will see pulses of light, when the strobe light of the particle points in his direction.

(b) The pulses have width

$$\Delta t \sim \frac{R_o/c}{\gamma^3} \tag{13.35}$$

You should be able to explain this result. Specifically, the light is *formed* at the source over a time, $\Delta T \simeq \frac{R_o/c}{\gamma}$, since the angular velocity of the source is R_o/c and the angular width of the particles radiation cone is $1/\gamma$. Then using the relation between *formation* time and *observation time*, Eq. (13.10), we find Δt.

The frequency width $\Delta\omega \sim 1/\Delta t$

$$\Delta\omega \sim \frac{\gamma^3}{R_o/c}$$

(c) The frequency spectrum for circular motion is derived by evaluating the integrals in Eq. (13.17) for circular motion. This is done in we evaluated this in the limit where the pulses are very narrow. The fourier spectrum of a single pulse is expressed in the following form

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{c}\gamma^2 F\left(\frac{\omega}{\omega_*}, \gamma\theta\right) \tag{13.36}$$

where

$$\omega_* = \frac{3c\gamma^3}{R_o} \tag{13.37}$$

where $F(x, y)$ is a dimensionless order one function of x, y. You should understand the qualitative features of the spectrum, and how these qualitative features are encoded in a formula like Eq. (13.36)

We record the result of integrating Eq. (13.17) for a single pulse

$$(2\pi)\frac{dW}{d\omega d\Omega} = \frac{3}{4}\frac{q^2}{\pi^2 c}\gamma^2 \left[\left(\frac{\omega}{\omega_*}\right)^{2/3}\left(\xi^{2/3}K_{2/3}(\xi)\right)^2 + \left(\frac{\omega}{\omega_*}\right)^{4/3}\left(\gamma\theta\xi^{1/3}K_{1/3}(\xi)\right)^2\right] \tag{13.38}$$

where

$$\xi = \frac{\omega}{\omega_*}(1 + (\gamma\theta)^2)^{3/2} \tag{13.39}$$

This specific formula might help you understand with the previous item.

(d) We Fourier analyzed a sequence of pulses in different contexts (e.g. a sequence of laser pulses or a sequence of synchrotron pulses). You should be able to show that the Fourier transform of n-pulses

$$E_n(\omega) = E_1(\omega)\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right) \tag{13.40}$$

where $E_1(\omega)$ is the Fourier transfrom of one pulse. This is used to show that the time average power radiatied into the m-th harmonic is

$$\frac{dP_m}{d\Omega} = \frac{1}{\mathcal{T}_o^2}|rE_1(\omega_m)|^2 \tag{13.41}$$

(e) Finally you should be able to prove the following identities, if

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o) \tag{13.42}$$

Then this function has a Fourier series representation:

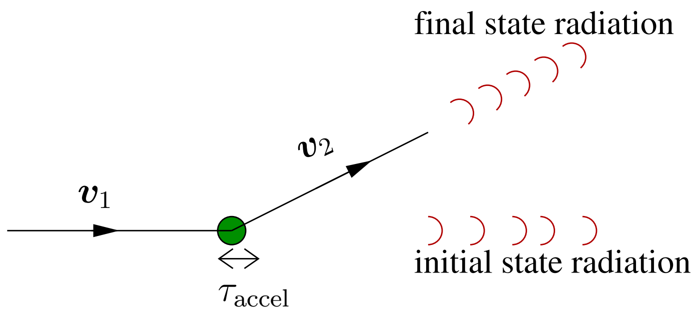
$$\Delta(t) = \frac{1}{\mathcal{T}_o} \sum_{m=-\infty}^{\infty} e^{-i\omega_m t} \tag{13.43}$$

with $\omega_m \equiv \frac{2\pi m}{\mathcal{T}_o}$. The Fourier transform of $\Delta(t)$ is

$$\Delta(\omega) = \sum_n e^{i\omega n \mathcal{T}_o} = \frac{2\pi}{\mathcal{T}_o} \sum_m \delta(\omega - \omega_m) \tag{13.44}$$

13.4 Bremsstrahlung

(a) During a collsion of charged particles, the scattered charged particles is rapidly accelerated over a short time period $\tau_{\rm accel}$, from $\boldsymbol{v}_1$ to $\boldsymbol{v}_2$. This causes radiation



(b) Evaluating the integrals in Eq. (13.17) or Eq. (13.17b), we find that the radiated energy spectrum is:

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2 c} \left| \frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}_2}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_2} - \frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}_1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_1} \right|^2 \tag{13.45}$$

The $\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{v}$ gives you the electric field, and the result is squared. One could also use the magnetic field

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2}{16\pi^2 c} \left| \frac{\boldsymbol{n}\times\boldsymbol{\beta}_2}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_2} - \frac{\boldsymbol{n}\times\boldsymbol{\beta}_1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_1} \right|^2 \tag{13.46}$$

(c) Much can be said about this important result:

i) It is independent of frequency. Thus it would seem that $\int_0^\infty d\omega \frac{dI}{d\omega d\Omega} \to \infty$. In practice the energy (photon) spectrum will agree with Eq. (13.45), until the photon energy is comparable to the energy of the particles. Or until the formation time of the radiation $\Delta T \sim \frac{1}{\omega(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}$ becomes comparable to the time scale of acceleration, $\tau_{\rm accel}$. For ultra-relativistic particles this means that:

$$\omega_{\rm max} \sim \frac{\gamma^2}{\tau_{\rm accel}(1+(\gamma\theta)^2)}$$

ii) Since the energy spectrum is independent of frequency the number of soft photons is divergent

$$\frac{dN}{d\omega} = \frac{1}{\hbar\omega}\frac{dI}{d\omega} \propto \frac{\alpha}{\omega} \tag{13.47}$$

where $\alpha \simeq q^2/(4\pi\hbar c) \simeq 1/137$ for an electron.

iii) For very relativistic particles the radiation is strongly peaked in either the direction of $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$, see figure. For very relativistic particles, $\gamma \to \infty$, you should be able to show that the number of photons per frequency interval, per angle (measured with respect to $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$) is approximately

$$dN \simeq \frac{2\alpha}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\theta} \tag{13.48}$$

Here θ is measured with respect either the $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ axes and is assumed to be small but large compared to $1/\gamma$: $\frac{1}{\gamma} \ll \theta \ll 1$. The fine structure constant is $\alpha = q^2/(4\pi\hbar c) \simeq 1/137$ for an electron. Thus we see that soft photons are logarithmically distributed in angle and in frequency.

14 Scattering

We formulated the scattering problem. In this case incoming light induces currents in the object, which in turn create a radiation field. We will work with small objects and weak scattering where the effect of the induced radiation fields can be neglected in determining the currents. The external incoming field will induce acceleration in the case of light-electron scattering, or induce time-dependent dipole moments (i.e. currents) in the case of light scattering off a sphere.

(a) The Electric field can be written

$$\boldsymbol{E} = \boldsymbol{E}_{\rm inc} + \boldsymbol{E}_{\rm scat} \tag{14.1}$$

where

$$\boldsymbol{E}_{\rm inc}(t, \boldsymbol{r}) = E_o\, \boldsymbol{\epsilon}_o e^{ikz - i\omega t} \tag{14.2}$$

while the scattered field falls of as $1/r$

$$\boldsymbol{E}_{\rm scat}(t, \boldsymbol{r}) \to \boldsymbol{C}(\boldsymbol{k}) \frac{e^{ikr - i\omega t}}{r} \tag{14.3}$$

$\boldsymbol{E}_{\rm scat}$ (in the far field) might as well be called $\boldsymbol{E}_{\rm rad}$. The constant is proportional for E_o for linear response and so the far field of the scattered field is written in terms of the scattering amplitude, $\boldsymbol{f}(\boldsymbol{k})$.

$$\boldsymbol{E}_{\rm scat}(t, \boldsymbol{r}) \to E_o \boldsymbol{f}(\boldsymbol{k}) \frac{e^{ikr - i\omega t}}{r} \tag{14.4}$$

(b) We will follow the following notation for harmonic fields. We write E_ω to notate the thing in front of $e^{-i\omega t}$

$$\boldsymbol{E}(t) = \boldsymbol{E}_\omega e^{-i\omega t} \tag{14.5}$$

Since writing $\boldsymbol{E}_{\omega,\rm scat}(\boldsymbol{r})$ gets old fast, we will just write $E_{\rm scat}(\boldsymbol{r})$ or simply $E_{\rm scat}$ without anything to mean $E_{\omega,\rm scat}(\boldsymbol{r})$ when clear from context

(c) The radiation field $\boldsymbol{E}_{\rm scat}$ can be decomposed into polarizations

$$\boldsymbol{E}_{\rm scat} = E_1 \boldsymbol{\epsilon}_1 + E_2 \boldsymbol{\epsilon}_2 \tag{14.6}$$

Using the orthogonality of the polarization vectors

$$\boldsymbol{\epsilon}_a^* \cdot \boldsymbol{\epsilon}_b = \delta_{ab}\,, \tag{14.7}$$

we have, *e.g.*

$$E_1 = \boldsymbol{\epsilon}_1^* \cdot \boldsymbol{E}_{\rm scat} \qquad E_2 = \boldsymbol{\epsilon}_2^* \cdot \boldsymbol{E}_{\rm scat}\,. \tag{14.8}$$

The time averaged power radiated per solid angle with polarization $\boldsymbol{\epsilon}_1$ is

$$\overline{\frac{dP}{d\Omega}(\boldsymbol{\epsilon}_1; \boldsymbol{\epsilon}_o)} = \frac{c}{2} |r\ \boldsymbol{\epsilon}_1^* \cdot \boldsymbol{E}_{\rm scat}|^2 \tag{14.9}$$

and similarly for $\boldsymbol{\epsilon}_2$. This will in general depend on the incoming polarization, $\boldsymbol{\epsilon}_o$, of the light.

(d) The cross section is the time averaged radiated power divided by the (time-averaged) input flux

$$\frac{d\sigma(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}_o)}{d\Omega} = \frac{\overline{\frac{dP}{d\Omega}(\boldsymbol{\epsilon}_1; \boldsymbol{\epsilon}_o)}}{\frac{c}{2}|E_o|^2} = |\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{f}(\boldsymbol{k})|^2 \tag{14.10}$$

Long Wavelength Scattering

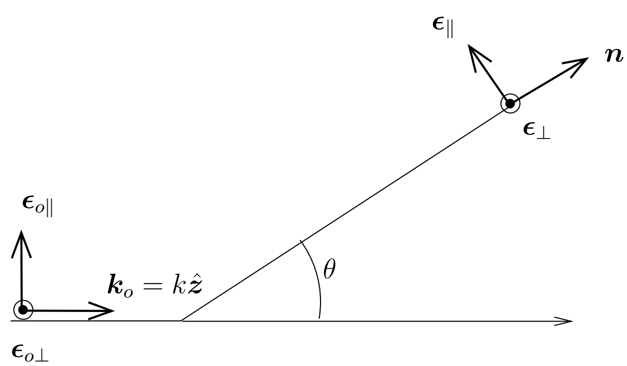
(a) We studied Thomson scattering (light-electron scatering) and found that the cross section was proportional to the classical electron radius squared

$$\sigma_T = \frac{8\pi}{3} r_e^2 \qquad r_e^2 = \left(\frac{q^2}{4\pi m c^2}\right)^2 \tag{14.11}$$

You should feel comfortable deriving this result and estimating the answer without looking up numbers. To derive the result compute the acceleration, and then compute the radiated electric field using Larmour type results (see Sect. (11.2) and Eq. (11.25) in particular). With the radiated field you can compute the power-radiated per solid angle with a given frequency.

(b) We also studied dipole scattering were we found that the cross section increases as ω^4. You should feel comfortable deriving this result. To derive the result you determine the induced dipole moment (electric, or magnetic, or both) in the applied field, and then use this induced dipole moment (which is oscillating) to compute the radiated field (see Eq. (11.33) and Eq. (11.42))

(c) The cross section for polarized scattering is found by considering the following picture:



So there are four cases depending on whether the incoming and outgoing polarizations are parallel or perpendicular to the scattering plane. For example, the cross section to produce light of polarization $\epsilon_{\perp}$ by un-polarized light (50% $\epsilon_{o\perp}$ and 50% $\epsilon_{o\parallel}$) is

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma(\epsilon_{\perp};\epsilon_{o\perp})}{d\Omega} + \frac{d\sigma(\epsilon_{\perp};\epsilon_{o\parallel})}{d\Omega}\right] \tag{14.12}$$

Born Approximation

(a) We showed that the scattering amplitude and current can be expressed in terms of the induced current. The cross section to produce light of any polarization is the square of the scattering amplitude

$$\frac{d\sigma}{d\Omega} = |\boldsymbol{f}(\boldsymbol{k})|^2 = \frac{k^2}{16\pi^2 E_o^2}\left|\boldsymbol{n}\times\int d^3\boldsymbol{r}_o \frac{\boldsymbol{J}_\omega(\boldsymbol{r}_o)}{c} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_o}\right|^2 . \tag{14.13}$$

This is just a rewriting of Eq. (11.60) using the definitions used in scattering. In the scattering problem we must also determine the current.

(b) In a Born approximation, the current in a dielectric medium is determined only by the incoming electric field, since the scattered field is small

$$\boldsymbol{J}_\omega(\boldsymbol{r}) = -i\omega\chi(\omega,\boldsymbol{r})\,\boldsymbol{E}_{\omega,\rm inc}(\boldsymbol{r})\,, \tag{14.14}$$

where

$$\boldsymbol{E}_{\omega,\rm inc}(\boldsymbol{r}) = E_o\boldsymbol{\epsilon}_o e^{i\boldsymbol{k}_o\cdot\boldsymbol{r}_o} \qquad \text{with} \qquad \boldsymbol{k}_o \equiv k\hat{\mathbf{z}}\,. \tag{14.15}$$

So the cross section in this approximation is

$$\frac{d\sigma}{d\Omega} = \left(\frac{k^2}{4\pi}\right)^2 |\boldsymbol{n} \times \boldsymbol{\epsilon}_o|^2 \left| \int_V d^3\boldsymbol{r}_o \, \chi(\omega, \boldsymbol{r}_o) e^{i(\boldsymbol{k}-\boldsymbol{k}_o)\cdot\boldsymbol{r}_o} \right|^2 . \tag{14.16}$$

A Heavside Lorenz (HL) Units

A.1 MKS to HL Units

- The HL Maxwell Equations follow from the MKS maxwell equations by defining

$$\boldsymbol{E}_{HL} = \sqrt{\epsilon_o}\boldsymbol{E}_{MKS} \qquad \boldsymbol{B}_{HL} = \frac{\boldsymbol{B}_{MKS}}{\sqrt{\mu_o}} \tag{A.1}$$

$$\rho_{HL} = \frac{\rho_{MKS}}{\sqrt{\epsilon}} \qquad \frac{\boldsymbol{j}_{HL}}{c} = \sqrt{\mu_o}\boldsymbol{j}_{MKS} \tag{A.2}$$

  and using $c = 1/\sqrt{\epsilon_o\mu_o}$

- To convert from MKS to HL set $\epsilon_o = 1$ (and thus $\mu_o = 1/c^2$, $\sqrt{\mu_o} = 1/c$) and use this table

Quantity	$\epsilon_o = 1$ relation
B-field	$c\boldsymbol{B}_{MKS} = B_{HL}$
A-field	$c\boldsymbol{A}_{MKS} = \boldsymbol{A}_{HL}$
magnetic dipole moment	$\frac{m_{MKS}}{c} = m_{HL}$
magnetization	$\frac{M_{MKS}}{c} = M_{HL}$
induction	$\frac{H_{MKS}}{c} = H_{HL}$
permeability	$\mu_{MKS}/\mu_o = \mu_{HL}$
permitivity	$\epsilon_{MKS}/\epsilon_o = \epsilon_{HL}$

In each of these examples the $\Longrightarrow$ indicates that I have set $\epsilon_o = 1$, so $\mu_o = 1/c^2$ when $\epsilon_o = 1$.

Example: the magnetic potential energy

$$U_B = \frac{1}{2}\frac{B_{MKS}^2}{\mu_o} \Longrightarrow \frac{1}{2}(cB_{MKS})^2 = \frac{1}{2}B_{HL}^2 \tag{A.3}$$

Example: The poynting vector

$$S = \frac{1}{\mu_o}\boldsymbol{E}_{MKS} \times \boldsymbol{B}_{MKS} \Longrightarrow c\boldsymbol{E}_{MKS} \times (c\boldsymbol{B}_{MKS}) = c\boldsymbol{E}_{HL} \times \boldsymbol{B}_{HL} \tag{A.4}$$

Example: The force law

$$\boldsymbol{F} = q_{MKS}(\boldsymbol{E}_{MKS} + \boldsymbol{v} \times \boldsymbol{B}_{MKS}) \Longrightarrow q_{MKS}(\boldsymbol{E}_{MKS} + \frac{\boldsymbol{v}}{c} \times c\boldsymbol{B}_{MKS}) = q_{HL}(\boldsymbol{E}_{HL} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}_{HL}) \tag{A.5}$$

Example: The magnetic energy of a dipole

$$U = -\boldsymbol{m}_{MKS} \cdot \boldsymbol{B}_{MKS} \Longrightarrow -\frac{\boldsymbol{m}_{MKS}}{c}(c\boldsymbol{B}_{MKS}) = -\boldsymbol{m}_{HL} \cdot \boldsymbol{B}_{HL} \tag{A.6}$$

Example: The Magnetic energy in matter

$$U = \frac{1}{2}\boldsymbol{B}_{MKS} \cdot \boldsymbol{H}_{MKS} \Longrightarrow \frac{1}{2}c\boldsymbol{B}_{MKS}\frac{\boldsymbol{H}_{MKS}}{c} = \frac{1}{2}\boldsymbol{B}_{HL} \cdot \boldsymbol{H}_{HL} \tag{A.7}$$

Example: Consistency of definition of $\boldsymbol{H}$

$$H_{MKS} = \frac{1}{\mu_o} B_{MKS} - M_{MKS} \Longrightarrow H_{MKS} = c^2 B_{MKS} - M_{MKS} \quad \text{or} \quad H_{HL} = B_{HL} - M_{HL} \tag{A.8}$$

The last step follows by dividing both sides by c.

A.2 HL to MKS

- The relation between charges and and currents in the HL and MKS units are

$$Q_{HL} = \frac{Q_{MKS}}{\sqrt{\epsilon}} \quad \rightarrow \quad \frac{1}{\sqrt{\epsilon_o}}\,(1\,\mu\text{C}) = 0.336\,\sqrt{N \cdot m^2} \tag{A.9}$$

$$\frac{I_{HL}}{c} = \frac{I_{MKS}}{\sqrt{\epsilon} c} = \sqrt{\mu_o} I_{MKS} \quad \rightarrow \quad \sqrt{\mu_o}(1\,\text{amp}) = 0.00112\,\sqrt{N \cdot m^2} \tag{A.10}$$

- The relation between Field strengths and is

$$E_{HL} = \sqrt{\epsilon_o}\,E_{MKS} \quad \rightarrow \quad \sqrt{\epsilon_o}\,(1\,\text{kV/cm}) = 0.2975\,\sqrt{N/m^2} \tag{A.11}$$

$$B_{HL} = \sqrt{\epsilon_o}\,(cB_{MKS}) = \frac{1}{\sqrt{\mu_o}} B_{MKS} \quad \rightarrow \quad \frac{1}{\sqrt{\mu_o}}(1\,\text{Tesla}) = 892.062\,\sqrt{N/m^2} \tag{A.12}$$

B Scalars, Vectors, Tensors

(a) We will use the Einstein summation convention

$$\boldsymbol{V} = V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + V^3\boldsymbol{e}_3 = V^i\boldsymbol{e}_i \tag{B.1}$$

Here repeated indices are implicitly summed from $i = 1 \ldots 3$, where $1, 2, 3 = x, y, z$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors in the x, y, z directions.

(b) Under a rotation of coordinates the coordinates change in the following way

$$x^i = R^i_{\ j} x^j \,. \tag{B.2}$$

where R we think of as a rotation matrix, where i labels the rows of R and j labels the columns of R.

(c) Scalars, vectors and tensors are defined by how there components transform

$$S \to \underline{S} = S \,, \tag{B.3}$$

$$V^i \to \underline{V}^i = R^i_{\ j} V^j \,, \tag{B.4}$$

$$T^{ij} \to \underline{T}^{ij} = R^i_{\ \ell} R^j_{\ m} T^{\ell m} \,. \tag{B.5}$$

We think of upper indices (contravariant indices) as row labels, and lower indices (covariant indicies) as column labels. Thus V^i is thought of as column vector

$$V^i \leftrightarrow \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{B.6}$$

labelled by V^1, V^2, V^3 – the first row entry, the second row entry, the third row entry. Contravariant means "opposite to coordinate vectors" $\boldsymbol{e}_i$ (see next item)

(d) Under a rotation of coordinates the basis vectors also transform with

$$\underline{\boldsymbol{e}}_i \to \underline{\boldsymbol{e}}_i (R^{-1})^i_{\ j} \tag{B.7}$$

This transformation rule is how the lower (or covariant) vectors transform. The covariant components of a vector V_i transform as

$$(\underline{V}_1 \underline{V}_2 \underline{V}_3) = (V_1 V_2 V_3)\Big(R^{-1}\Big) \,. \tag{B.8}$$

covariant means "the same as coordinate vectors", *i.e.* with R^{-1} but as a row.

(e) Since $R^{-1} = R^T$ there is no need to distinguish covariant and contravariant indices for rotations. This is not the case for more general groups.

(f) With this notation the vectors and tensors (which are physical objects)

$$\underline{\boldsymbol{V}} = \underline{V}^i \underline{\boldsymbol{e}}_i \quad = \quad V^i \boldsymbol{e}_i = \boldsymbol{V} \tag{B.9}$$

$$\underline{\boldsymbol{T}} = \underline{T}^{ij} \underline{\boldsymbol{e}}_i \underline{\boldsymbol{e}}_j \quad = \quad T^{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{T} \tag{B.10}$$

are invariant under rotations, but the components and basis vectors change.

(g) Vector and tensor components can be raised and lowered with δ^{ij} which forms the identity matrix,

$$\delta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.11}$$

i.e.

$$V^i = \delta^{ij} V_j \tag{B.12}$$

We note various trivia

$$\delta^i_{\ i} = 3 \qquad \delta_{ij}\delta^{ij} = 3 \quad \delta_{ij}\delta^{jk} = \delta^k_i \tag{B.13}$$

(h) The epsilon tensor ϵ^{ijk} is

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} \pm 1 & \text{for } i,j,k \text{ an even/odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases} \tag{B.14}$$

For example, $\epsilon^{123} = \epsilon^{312} = \epsilon^{231} = 1 = \epsilon_{123} = 1$ while $\epsilon^{213} = -\epsilon^{123} = -1$.

i) The epsilon tensor is useful for simplifying cross products

$$(\boldsymbol{a} \times \boldsymbol{b})^i = \epsilon^{ijk} a_j b_k \tag{B.15}$$

ii) A useful identity is

$$\epsilon^{ijk}\epsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl} \tag{B.16}$$

which can be used to deduce the "b(ac) - (ab)c" rule for cross products

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{B.17}$$

iii) The "b(ac) - (ab)c" rule arises a lot in this course and is essential to deriving the wave equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \tag{B.18}$$

and to identifying the transverse pieces of a vector. For instance the component of a vector $\boldsymbol{v}$, transverse to a unit vector $\boldsymbol{n}$, is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{v}) = \boldsymbol{v}_T = -(\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} + \boldsymbol{v} \tag{B.19}$$

(i) Derivatives work the same way. $\partial_i \equiv \frac{\partial}{\partial x^i}$. With this notation we have

$$\nabla \cdot \boldsymbol{E} = \partial_i E^i \tag{B.20}$$

$$(\nabla \times \boldsymbol{E})^i = \epsilon^{ijk} \partial_j E_k \tag{B.21}$$

$$(\nabla \phi)_i = \partial_i \phi \tag{B.22}$$

$$(\nabla^2 \phi) = \partial_i \partial^i \phi \tag{B.23}$$

$$\tag{B.24}$$

and expressions like

$$\partial_i x^j = \delta^j_i \qquad \partial_i x^i = d = 3 \tag{B.25}$$

(j) A general second rank tensor T^{ij} is decomposed into its irreducible components as

$$T^{ij} = \mathring{T}^{ij}_S + \epsilon^{ijk} V_k + \tfrac{1}{3} T^\ell_{\ \ell} \delta^{ij} \tag{B.26}$$

where $\mathring{T}^{ij}_S = \frac{1}{2}(T^{ij} + T^{ji} - \frac{2}{3} T^\ell_{\ \ell}\delta^{ij})$ is a *symmetric-traceless* component of T^{ij} and V_k is a vector associated with the antisymmetric part of T^{ij}, $V_k = \frac{1}{2}\epsilon_{k\ell m} T^{\ell m}$.

(k) We will discussed how to reduce a tensor integral into a set of scalar integrals later in this course, e.g.

$$\int d^3\boldsymbol{x}\; x^i x^j x^\ell x^m\, f(x) = \left[\frac{4\pi}{15}\int_0^\infty dx\, x^6 f(x)\right] \left(\delta^{ij}\delta^{\ell m} + \delta^{i\ell}\delta^{jm} + \delta^{im}\delta^{j\ell}\right) \tag{B.27}$$

Here $x = |\boldsymbol{x}|$ denotes the norm of the vector $\boldsymbol{x}$. Thus, $f(x)$ denotes a function of the radius, $f(\sqrt{x_1^2 + x_2^2 + x_3^2})$.

C Fourier Series and other eigenfunction expansions

We will often expand a function in a complete set of eigen-functions. Many of these eigen-functions are traditionally not normalized. Using the quantum mechanics notation we have

$$|F\rangle = \sum_n F_n \frac{1}{C_n} |n\rangle \qquad \text{where} \qquad F_n = \langle n|F\rangle \qquad \text{and} \qquad \langle n_1|n_2\rangle = C_{n_1}\delta_{n_1 n_2} \tag{C.1}$$

The inner product (in 1D) will have the form

$$\langle F|G\rangle = \int_a^b \mathrm{d}x\, r(x)\, F^*(x)G(x) \tag{C.2}$$

where $r(x) > 0$ is a positive definite weight function. In what follows we show the eigen-function in square brackets

More prosaically these conditions read

$$F(x) = \sum_n F_n \frac{1}{C_n} [\psi_n(x)] \,, \tag{C.3}$$

$$F_n = \int \mathrm{d}x\, r(x)\; [\psi_n^*(x)]\; F(x)\,, \tag{C.4}$$

$$\int_a^b \mathrm{d}x\, r(x)\; \left[\psi_{n_1}^*(x)\right] [\psi_{n_2}(x)] = C_{n_1}\delta_{n_1 n_2}\,. \tag{C.5}$$

We require that the functions are complete (in the space of functions which satisfy the same boundary conditions as F)

$$\sum_n \frac{1}{C_n} |n\rangle \langle n| = I\,, \qquad \text{or} \qquad \sum_n \frac{1}{C_n} \psi_n(x)\psi_n^*(x') = \frac{1}{r(x)}\delta(x-x')\,. \tag{C.6}$$

We give a list of the most common eigen functions:

(a) A periodic function $F(x)$ with period L is expandable in a Fourier series. Defining $k_n = 2\pi n/L$ with n integer:

$$F(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \left[e^{ik_n x}\right]\; F_n \tag{C.7a}$$

$$F_n = \int_0^L \mathrm{d}x\; \left[e^{-ik_n x}\right]\; F(x) \tag{C.7b}$$

$$\int_0^L \mathrm{d}x\; [e^{-ik_n x}]\, [e^{ik_{n'} x}] = L\,\delta_{nn'} \tag{C.7c}$$

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \sum_m \delta(x - x' + mL) \tag{C.7d}$$

The last identity is known as the Poisson summation formula. Eq. (C.7d) can be rewritten by fourier transforming $\int dx\, e^{ikx}$ both sides

$$\frac{1}{L}\sum_n 2\pi\delta\left(k - \frac{2\pi n}{L}\right) = \sum_m e^{ikmL} \,. \tag{C.8}$$

(b) A square integrable function in one dimension has a Fourier transform

$$F(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikz}\right] F(k) \tag{C.9}$$

$$F(k) = \int_{-\infty}^{\infty} dz \left[e^{-ikz}\right] F(z) \tag{C.10}$$

$$\int_{-\infty}^{\infty} dz\, e^{-iz(k-k')} = 2\pi\delta(k-k') \tag{C.11}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(z-z')} = \delta(z-z') \tag{C.12}$$

You should feel comfortable deriving the Fourier transform from its discrete counter part Eq. (C.7) by taking $L \to \infty$.

(c) A regular function on the sphere (θ, ϕ) can be expanded in spherical harmonics

$$F(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left[Y_{\ell m}(\theta,\phi)\right] F_{\ell m} \tag{C.13}$$

$$F_{\ell m} = \int d\Omega \, \left[Y_{\ell m}^*(\theta,\phi)\right] F(\theta,\phi) \tag{C.14}$$

$$\int d\Omega \left[Y_{\ell m}^*(\theta,\phi)\right]\left[Y_{\ell' m'}(\theta,\phi)\right] = \delta_{\ell\ell'}\delta_{mm'} \tag{C.15}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left[Y_{\ell m}(\theta,\phi)\right]\left[Y_{\ell m}^*(\theta',\phi')\right] = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi') \tag{C.16}$$

(d) When expanding a function on the sphere with azimuthal symmetry, the full set of $Y_{\ell m}$ is not needed. Only $Y_{\ell 0}$ is needed. $Y_{\ell 0}$ is related to the Legendre Polynomials. We note that

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \tag{C.17}$$

A function $F(\cos\theta)$ between $\cos\theta = -1$ and $\cos\theta = 1$ can be expanded in Legendre Polynomials.

$$F(\cos\theta) = \sum_{\ell=0}^{\infty} F_\ell \, \frac{2\ell+1}{2} \left[P_\ell(\cos\theta)\right] \tag{C.18}$$

$$F_\ell = \int_{-1}^{-1} d(\cos\theta) \, \left[P_\ell(\cos\theta)\right] F(\cos\theta) \tag{C.19}$$

$$\int_{-1}^{1} d(\cos\theta)\left[P_\ell(\cos\theta)\right]\left[P_{\ell'}(\cos\theta)\right] = \frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{C.20}$$

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\left[P_\ell(\cos\theta)\right]\left[P_\ell(\cos\theta')\right] = \delta(\cos\theta - \cos\theta') \tag{C.21}$$

(e) A function, $F(\rho)$ on the half line $\rho = [0, \infty]$, which vanishes like ρ^m as $\rho \to 0$ can be expanded in Bessel functions. This is known as a Hankel transform and arises in cylindrical coordinates

$$F(\rho) = \int_0^\infty kdk \;\; [J_m(k\rho)] \;\; F_m(k) \tag{C.22}$$

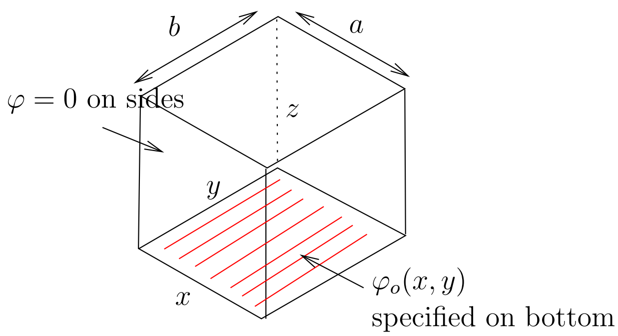
$$F_m(k) = \int_0^\infty \rho d\rho \;\; [J_m(k\rho)] \;\; F(\rho) \tag{C.23}$$

$$\int_0^\infty \rho d\rho \;\; [J_m(\rho k)] \, [J_m(\rho k')] = \frac{1}{k}\delta(k - k') \tag{C.24}$$

$$\int_0^\infty kdk \;\; [J_m(\rho k)] \, [J_m(\rho' k)] = \frac{1}{\rho}\delta(\rho - \rho') \tag{C.25}$$

D Separation of Variables

D.1 Cartesian coordinates



(a) Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0 \tag{D.1}$$

(b) Eigen fucntions along boundary vanishing at $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad n = 1\ldots\infty \qquad m = 1\ldots\infty$$

(c) Orthogonality

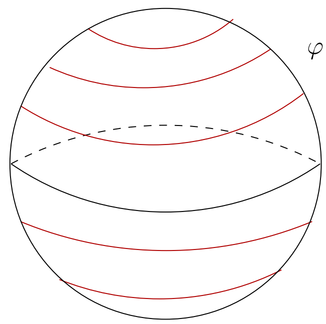
$$\int_0^a dx \int_0^b dy\, \psi_{nm}\,\psi_{n'm'} = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\delta_{nn'}\delta_{mm'}$$

(d) Solution

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}e^{-\gamma_{nm}z} + B_{nm}e^{+\gamma_{nm}z}\right]\psi_{nm}(x, y) \tag{D.2}$$

where $\gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2}$

D.2 Spherical coordinates



(a) Laplacian

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Phi = 0 \tag{D.3}$$

(b) Eigen fucntions along boundary θ, ϕ, regular at $\theta = 0$ and π, 2π periodic in ϕ

$$\psi_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, \phi) \qquad \ell = 0 \ldots \infty \quad m = -\ell \ldots \ell$$

(c) Orthogonality:

$$\int d\Omega\, Y^*_{\ell m}(\theta, \phi)\, Y_{\ell' m'}(\theta, \phi) = \delta_{\ell\ell'}\delta_{mm'}$$

(d) Solution

$$\Phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right]Y_{\ell m} \tag{D.4}$$

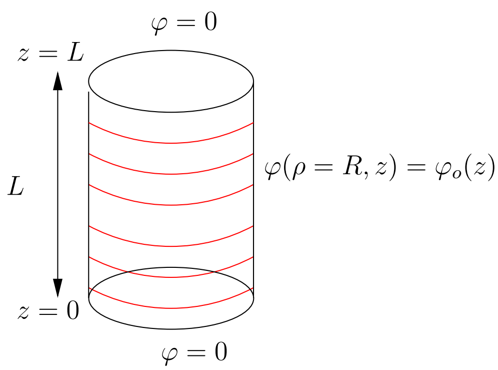
(e) When there is no azimuthal dependence things simplify to

$$\Phi = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta) \tag{D.5}$$

where $P_\ell(\cos\theta)$ is the legendre polynomial, which up to a normalization if $Y_{\ell 0}(\theta, \phi)$, satisfying the orthogonality

$$\int_{-1}^{1} d(\cos\theta)P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

D.3 Cylindrical Boundary: z, ϕ are the boundary.



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0 \tag{D.6}$$

(b) Eigenfunctions along boundary z, ϕ vanishing at $z = 0$ and $z = L$ and 2π periodic in ϕ

$$\psi_{nm}(z,\phi) = \sin(k_n z)\, e^{im\phi} \qquad k_n \equiv \frac{n\pi}{L} \qquad n = 1\ldots\infty \;\; m = -\infty\ldots\infty$$

(c) Orthogonality:

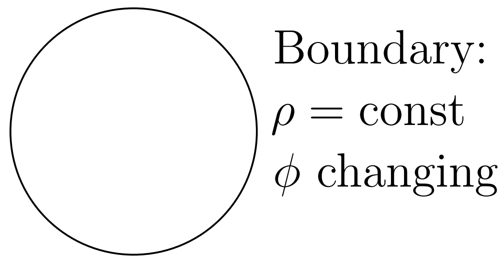
$$\int_0^L dz \int_0^{2\pi} \psi_{nm}(z,\phi)\, \psi_{nm}(z,\phi) = \frac{L}{2}\,(2\pi)\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{nm} I_m(k_n\rho) + B_{nm} K_m(k_n\rho)\right]\psi_{nm}(z,\phi) \tag{D.7}$$

Here $I_\nu(x)$ and $K_\nu(x)$ is the modified bessel function of the first and second kinds. Note that $K_{-m}(x) = K_m(x)$ and $I_{-m}(x)$

D.4 2D cylindrical coordinates



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\Phi = 0 \tag{D.8}$$

(b) Eigenfunctions along boundary ϕ: 2π periodic in ϕ

$$\psi_m(\phi) = e^{im\phi} \qquad m = -\infty \ldots \infty$$

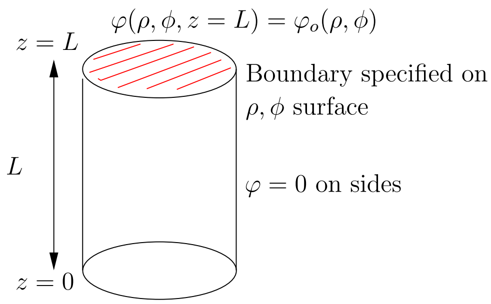
(c) Orthogonality

$$\int_0^{2\pi} \psi_m^*(\phi)\psi_{m'}(\phi) = 2\pi\delta_{mm'} \tag{D.9}$$

(d) Solution

$$\Phi = A_0 + B_0 \ln\rho + \sum_{m=-\infty}^{\infty}\left(A_m\rho^{|m|} + \frac{B_m}{\rho^{-|m|}}\right)\psi_m$$

D.5 Cylindrical Boundary: ρ, ϕ are the boundary



(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0 \tag{D.10}$$

(b) Eigenfunctions along boundary ρ, ϕ vanishing at $\rho = R$ and regular at $\rho = 0$, 2π periodic in ϕ:

$$\psi_{mn}(\rho, \phi) = J_m\left(k_{mn}\rho\right) e^{im\phi} \qquad n = 1\ldots\infty \;\; m = -\infty\ldots\infty$$

Here:

$$k_{mn} = \frac{x_{mn}}{R} \tag{D.11}$$

where x_{mn} is the n-th zero of the m-th Bessel function, *e.g.* the zeros of $J_0(x)$ are

$$(x_{01}, x_{02}, x_{03}) = 2.40483,\; 5.52008,\; 8.65373 \tag{D.12}$$

These are given by $x_{mn} = \mathrm{BesselZeroJ[m, n]}$ in Mathematica. Note also that $J_{-m}(x) = J_m(x)$

(c) Orthogonality:

$$\int_0^R \rho d\rho \int_0^{2\pi} \psi_{mn}(\rho, \phi)\, \psi_{mn}(\rho, \phi) = \left(\frac{R^2}{2}\left[J_{m+1}(k_{mn}R)\right]^2\right)(2\pi)\,\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\Phi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{mn}e^{-k_{mn}z} + B_{nm}e^{k_{mn}z}\right]\psi_{mn}(\rho, \phi) \tag{D.13}$$

D.6 Continuum Forms and Fourier and Hankel Transforms

In each case we are expanding two directions of the solution in a complete set of eigenfunctions

$$\langle x|F\rangle = \frac{1}{C_n}\sum_n \langle x|n\rangle \langle n|F\rangle \ , \tag{D.14}$$

and solving the laplace equation to find the dependence on the third direction.

(a) For the cartesian case when a and b go to infinity. The sum becomes an integral and the sum over n and m becomes a 2D fourier transform

$$\Phi = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \left[A(\boldsymbol{k}_\perp)e^{-k_\perp z} + B(\boldsymbol{k}_\perp)e^{k_\perp z}\right] \ .$$

We are using the fact that any function in the x, y plane (in particular the boundary condition $\Phi_o(x, y)$) can be expressed as a fourier transform pairs

$$F(x,y) \equiv \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \left[e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\right] F(k_x,k_y)\,, \tag{D.15}$$

$$F(k_x,k_y) \equiv \int d^2\boldsymbol{x}_\perp \left[e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\right] F(x,y)\,. \tag{D.16}$$

(b) For the cylindrical case when L goes to ∞, the sum over n becomes an integral yielding

$$\Phi = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi} \left[e^{i\kappa z}e^{im\phi}\right] \left[A(\kappa)I_m(|\kappa|\rho) + B(k)K_m(|\kappa|\rho)\right]$$

We are using the fact that any regular function of z and ϕ (in particular the boundary condition $\Phi_o(z, \phi)$) can be written in terms of its fourier components

$$F(z,\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi} \left[e^{i\kappa z}e^{im\phi}\right] F_m(\kappa) \tag{D.17}$$

$$F_m(\kappa) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left[e^{-i\kappa z}e^{-im\phi}\right] F(z,\phi) \tag{D.18}$$

(c) Finally for the second cylindrical case when the radius goes to infinity

$$\Phi = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk \left[J_m(k\rho)e^{im\phi}\right]\left[A(k)e^{-kz} + B(k)e^{kz}\right] \tag{D.19}$$

We are using the fact that any regular cylindrical function of ρ and ϕ (in particular the boundary condition $\Phi_o(\rho, \phi)$) can be written as *Hankel* transform

$$F(\rho,\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk \left[J_m(k\rho)e^{im\phi}\right] F_m(k) \tag{D.20}$$

$$F_m(k) = \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho \left[J_m(k\rho)e^{-im\phi}\right] F(\rho,\phi) \tag{D.21}$$

E Separated coordinates for magnetostatics

E.1 Overview

(a) The magnetostatic equations are complicated, and we refer to Wikipedia for the form of the vector Laplacian in various coordiante systems.

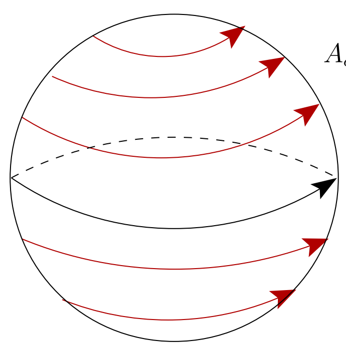
(b) For currents running up stricly up and down $A_z(x,y)$ the magneto static equations reduce to

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)A_z(x,y)=0 \tag{E.1}$$

which has the same form as $2D$ electrostatics. The appropriate separated solutions are given in Appendix D.4.

(c) For currents which are azimuthally symmetric $\boldsymbol{j}=j_\phi(r,\theta)\,\widehat{\boldsymbol{\phi}}$ we may either use spherical or cylindrical coordinates. The spherical case is discussed in Appendix E.2.

E.2 Spherical coordinates for magnetostatics



(a) The vector Laplacian for azimuthally symmetric currents, and the ansatz $\boldsymbol{A}=A_\phi(r,\theta)\,\widehat{\boldsymbol{\phi}}$ reads

$$\left[-\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\right]A_\phi(r,\theta)=0 \tag{E.2}$$

This is an appropriate equation only if the current takes a specific symmetric form

$$\boldsymbol{j}=j_\phi(r,\theta)\,\widehat{\boldsymbol{\phi}}$$

(b) The eigen fucntions are along the boundary direction θ, and are regular at $\theta=0$ and π. They are associated Legendre Polynomials with $m=1$

$$\psi_\ell(\theta)=P_\ell^1(\cos\theta)\qquad \ell=1\ldots\infty$$

The first few eigenfunctions are given here. Perhaps the most important fact is that they all are proportional to $\sin(\theta)$ guaranteeing regularity of $\boldsymbol{A} = A_\phi\,\widehat{\boldsymbol{\phi}}$ at $\theta = 0$ and π.

(c) Orthogonality:

$$\int_{-1}^{1} d(\cos\theta)\; P_\ell^1(\cos\theta)\; P_{\ell'}^1(\cos\theta) = \frac{2}{2\ell+1}\frac{(\ell+1)!}{(\ell-1)!}\delta_{\ell\ell'}$$

(d) Completeness

$$\sum_{\ell=1}^{\infty} \frac{2\ell+1}{2}\frac{(\ell-1)!}{(\ell+1)!} P_\ell^1(x)P_\ell^1(x') = \delta(x-x') \tag{E.3}$$

(e) Solution

$$A_\phi(r,\theta) = \sum_{\ell=1}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell^1(\cos\theta) \tag{E.4}$$