Constitutive Relation For Dielectric Media

- Now we will specify the currents in the medium. Usually we divide them into external (those explicitly specified/given) and medium currents which are specified by a constitutive eqn:

$$\vec{j} = \vec{j}_{mat} + \vec{j}_{ext}$$

- Once $\vec{j}$ is specified, we can use charge conservation to find ρ:

$$\partial_t \rho + \nabla \cdot \vec{j} = 0$$

(1) Treat j_{mat} as an expansion in $\vec{E}$ fields.
(2) Write down all possible terms allowed by symmetry
(3) Neglect $\vec{B}$-fields for now and ignore spatial derivatives (we will justify the neglect later)

- keep first order in $\vec{E}$

- also assume an isotropic medium

Typically if the medium as length scale $\ell_{\rm micro}$ and time scale $\tau_{\rm micro}$, the ratio scales as a typical velocity which is much less than light $\ell_{\rm micro}/\tau_{\rm micro} \sim v_{\rm micro} \ll c$. For light the length is L and the time is $T \sim L/c$. The spatial gradients are controlled by $\ell_{\rm micro}/L$, while the temporal gradients are controlled by $\tau_{\rm micro}/T$. The spatial gradients are thus smaller than there temporal counter parts by v/c

$$\frac{\ell_{\rm micro}}{L} \sim \frac{v_{\rm micro}}{c}\frac{\tau_{\rm micro}}{T} \ll \frac{\tau_{\rm micro}}{T}$$

- Then $\vec{j}$ must have the form

These break T-reversal

$$\vec{j}_{mat} = \sigma \vec{E} + \chi \partial_t \vec{E} + \sigma_2 \partial_t \partial_t \vec{E} + \chi_3 \partial_t^3 \vec{E}$$

$$+ \ldots$$

- If time reversal is a symmetry of the EOM then $\vec{E}$ is t-reverse even, while $\vec{j}$ is t-reversal odd.

Thus the terms σ and σ_2 break time reversal and describe dissipative processes (Ohms Law).

- Often σ is very small for insulators and can be discarded, for metals σ is not small and $\vec{j} = \sigma \vec{E}$ is the dominant term.

- Further,

Each higher order term is suppressed when the timescales of the E-field, T, are much longer than microscopic time-scales leading to $\vec{j}_{mat} = \chi \partial_t \vec{E}$ for insulators (dielectrics)

$$\vec{j}_{mat} = \sigma \vec{E} \quad \text{for conductors}$$

Ohm's law

Constituent Relations (Continued)

Reason. Dimensional Analysis:

Now we give a long winded version of why higher order derivatives are small.

$$[j] = \frac{q}{m^2}\frac{1}{S}$$

S = seconds

m = meter

$$[E] = \frac{q}{m^2}$$

q = charge

So:

$$[\chi] = 1$$

$$[\sigma_2] = S$$

$$[\chi_3] = S^2$$

$$[\sigma_4] = S^3$$

$$\vdots$$

Thus expect:

$$\chi \sim 1$$

in fact χ is of order unity for most materials

$$\sigma_2 \sim \tau_{micro} \quad \text{(or even less)}$$

$$\chi_3 \sim \tau_{micro}^2$$

$$\sigma_4 \sim \tau_{micro}^3$$

While for a macro-time scale $T \gg \tau_{micro}$:

$$\partial_t E \sim \frac{1}{T} E \quad \text{and} \quad \partial_t^2 E \sim \frac{1}{T^2} E \quad \ldots$$

Thus:

we are assuming insulators here

$$\vec{j} = \cancel{\sigma E} + \chi\,\partial_t E + \sigma_2\,\partial_t^2 E + \chi_3\,\partial_t^3 E$$

$$\sim 0 + \frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)\frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)^2 \frac{E}{T} + \ldots$$

Each higher term is suppressed by $\left(\frac{\tau_{micro}}{T}\right)$

Constituent Relation (Final)

Thus at lowest order in the gradient expansion

we are assuming insulators here

$$\vec{j} = \chi \partial_t \vec{E} \qquad\qquad \text{polarization vector}$$

$$\boxed{\vec{j} = \partial_t \vec{P}} \qquad\qquad \boxed{\vec{P} = \chi \vec{E}} \leftarrow \text{Linear isotropic media}$$

$\vec{j}$ is T-odd   T-even   T-even

So we can work out the charge density:

- From

$$\rho(\omega,k) = \frac{\vec{k}\cdot\vec{J}}{\omega} \quad \text{and} \quad \vec{j}(\omega,k) = -i\omega \vec{P} \Longleftrightarrow \vec{j} = \partial_t \vec{P}$$

Find $\rho(\omega,k) = -i\vec{k}\cdot\vec{P}$ or $\boxed{\rho = -\vec{\nabla}\cdot\vec{P}}$

- Or could have used coordinate space:

$$\partial_t \rho + \nabla\cdot j = 0$$

$$\partial_t \rho + \partial_i \partial_t P^i = 0$$

$$\partial_t (\rho + \partial_i P^i) = 0 \implies \boxed{\rho = -\partial_i P^i}$$

we could add a constant, but we are requiring overall neutrality here

Constituent Relation in EOM

With this we get the Eqs of motion:

$$\nabla \cdot E = \rho_{mat} + \rho_{ext} \qquad \nabla \times E = 0$$

$$\nabla \cdot E = -\nabla \cdot P + \rho_{ext}$$

Now

$$\nabla \cdot (\vec{E} + \vec{P}) = \rho_{ext}$$

with $\vec{P} = \chi \vec{E}$ for linear isotropic matter

$$\nabla \times E = 0$$

So define $\boxed{\vec{D} = \vec{E} + \vec{P}}$ and find

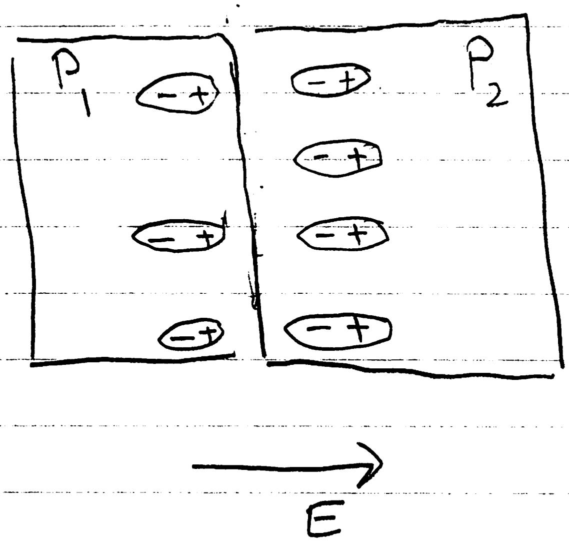
$$\boxed{\left.\begin{array}{l} \nabla \cdot \vec{D} = \rho_{ext} \\ \nabla \times E = 0 \end{array}\right\} \text{ eqs of macroscopic matter}}$$

Where $\vec{D} \equiv \vec{E} + \vec{P} \Rightarrow \underbrace{(1 + \chi)}_{\equiv \varepsilon} \vec{E}$ for a ~~linear medium~~ isotropic medium

linear relation

$$\varepsilon = 1 + \chi$$

The material charge at the interface



$P_2 > P_1$.

There is a net negative charge at the interface from the material

First lets caclute the surface charge.
For simplicity, set the external or "free" charge to zero at the interface:

We showed generally that

$$\vec{n} \cdot (\vec{E}_2 - \vec{E}_1) = \sigma = \overset{0 \text{ for simplicity}}{\cancel{\sigma_{ext}}} + \sigma_{mat}$$

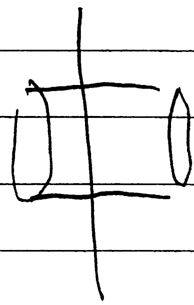
Then we have

$$\nabla \cdot E = \overset{0 \text{ simplicity}}{\cancel{\rho_{ext}}} + \rho_{mat}$$

$$\nabla \cdot E = -\nabla \cdot P$$

we showed this on the previous page

So from gauss Law



$$\vec{n} \cdot (\vec{E}_2 - \vec{E}_1) = -\vec{n} \cdot (\vec{P}_2 - \vec{P}_1)$$

$$\cancelto{0}{\sigma_{ext}} + \sigma_{mat} = -n \cdot (\vec{P}_2 - \vec{P}_1)$$

for simplicity

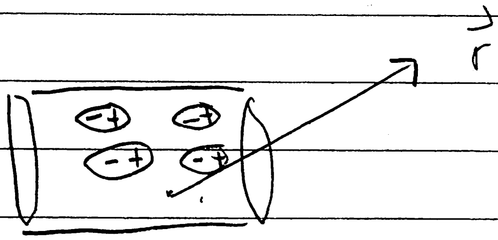
So

$$\boxed{\sigma_{mat} = -\vec{n} \cdot (\vec{P}_2 - \vec{P}_1)}$$

This is what we expect based on the dipole picture

Relation to the Dipole Picture (see also Jackson 4.3)

Consider a polarized object, and lets determine the potential at $\vec{r}$.



One could hope that the potential is given by a sum of dipole potentials.

The potential at $\vec{r}$ is

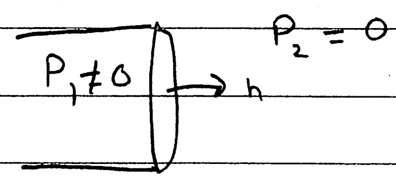
$$\varphi(\vec{r}) = \int_V d^3r_0 \, \frac{\rho_{mat}(r_0)}{4\pi|\vec{r}-\vec{r}_0|} + \int_S da \, \frac{\sigma_{mat}(x)}{4\pi|\vec{r}-\vec{x}|}$$

where $\vec{r}_0$ runs over the volume and $\vec{x}$ runs over the surface

using $\rho = -\nabla\cdot\vec{P}$ and

and $\sigma = -\vec{n}\cdot(\cancel{\vec{P}_2}^{\,0}-\vec{P}_1)$

$= \vec{n}\cdot\vec{P}_1$



Find

$$\varphi(r) = \int d^3r_0 \, \frac{-\partial_i P^i(r_0)}{4\pi|\vec{r}-\vec{r}_0|} + \int_S \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|}$$

integrate by parts:

$$\frac{\frac{\partial P^i}{\partial r_0^i}}{4\pi|\vec{r}-\vec{r}_0|} = \frac{\partial}{\partial r_0^i}\left(\frac{-P^i}{4\pi|\vec{r}-\vec{r}_0|}\right) + \frac{P^i(\vec{r}-\vec{r}_0)_i}{4\pi|\vec{r}-\vec{r}_0|^3}$$

Leading to:

$$\varphi(\vec{r}) = \int d^3r_0 \, \frac{\vec{P}\cdot(\vec{r}-\vec{r}_0)}{4\pi|\vec{r}-\vec{r}_0|^3} \cancel{- \int_x \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|} + \int_x \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|}}$$

This has the form of a dipole field from each volume element