4 Electric Fields in Matter

4.1 Parity and Time Reversal

(a) We discussed how fields transform under parity and time reversal. A useful table is

Quantity	Parity	Time Reversal
$\boldsymbol{r}(t)$	Odd	Even
$\boldsymbol{p}(t)$	Odd	Odd
$\mathrm{L} = \boldsymbol{r} \times \boldsymbol{p}$	Even	Odd
$\boldsymbol{F}$ =force	Odd	Even
Q = charge	Even	Even
ρ	Even	Even
$\boldsymbol{j}$	Odd	Odd
$\boldsymbol{E}$	Odd	Even
$\boldsymbol{B}$	Even	Odd

(b) In the table above the force is odd if parity is a symmetry of the theory. Similarly $\boldsymbol{j}$ is odd under time reversal only if time-reversal is a symmetry of the theory. In a dissipative media, $\boldsymbol{j}$ is not odd under time-reversal (though the microscopic currents are) and time-reversal is not a symmetry of macroscopic electrodynamics.

(c) For example, for a parity invariant theory, a solution to the maxwell equations $\boldsymbol{E}(t, \boldsymbol{x}), \boldsymbol{B}(t, \boldsymbol{x})$ determines a new solution to the Maxwell equations $\underline{E}(t, \boldsymbol{x}), \underline{\boldsymbol{B}}(t, \boldsymbol{x})$ can be found through inversion

$$\boldsymbol{E}(t, \boldsymbol{x}) \rightarrow \underline{\boldsymbol{E}}(t, \boldsymbol{x}) = -\boldsymbol{E}(t, -\boldsymbol{x}) \tag{4.1}$$

$$\boldsymbol{B}(t, \boldsymbol{x}) \rightarrow \underline{\boldsymbol{B}}(t, \boldsymbol{x}) = \boldsymbol{E}(t, -\boldsymbol{x}) \tag{4.2}$$

as specified by last two rows of the first column of the table

4.2 Electrostatics in Material

Basic setup

(a) In material we expand the medium currents $\boldsymbol{j}_{mat}$ in terms of a constitutive relation, fixing the currents in terms of the applied fields.

$$\boldsymbol{j}_{\mathrm{mat}} = [\text{ all possible combinations of the fields and their derivatives}] \tag{4.3}$$

We have added a subscript mat to indicate that the current is a medium current. There is also an external current $\boldsymbol{j}_{\mathrm{ext}}$ and charge density ρ_{ext}.

(b) When only uniform electric fields are applied, and the electric field is weak, and the medium is isotropic, the polarization current takes the form

$$\boldsymbol{j}_{\rm mat} = \sigma \boldsymbol{E} + \chi \partial_t \boldsymbol{E} + \dots \tag{4.4}$$

where the ellipses denote higher time derivatives of electric fields, which are suppressed by powers of t_{micro}/T_{macro} by dimensional analysis. For a conductor σ is non-zero. For a dielectric insulator σ is zero, and then the current takes the form

$$\boldsymbol{j}_b = \partial_t \boldsymbol{P} \tag{4.5}$$

- $\boldsymbol{P}$ is known as the polarization, and can be interpreted as the dipole moment per volume.
- We have worked with linear response for an isotropic medium where

$$\boldsymbol{P} = \chi \boldsymbol{E} \tag{4.6}$$

  This is most often what we will assume.

  For an anisotropic medium, χ is replaced by a susceptibility tensor

$$\boldsymbol{P}_i = \chi_{ij} \boldsymbol{E}^j \tag{4.7}$$

  For a nonlinear (isotropic) medium $\boldsymbol{P}$ one could try a non-linear vector function of $\boldsymbol{E}$,

$$\boldsymbol{P}(\boldsymbol{E}) \tag{4.8}$$

  defined by the low-frequency expansion of the current at zero wavenumber, but this is rather too simplistic for real ferro-electrics.

(c) Current conservation $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$ determines then that

$$\rho_{\rm mat} = -\nabla \cdot \boldsymbol{P} \tag{4.9}$$

(d) The electrostatic maxwell equations read

$$\nabla \cdot \boldsymbol{E} = \underbrace{-\nabla \cdot \boldsymbol{P}}_{\rho_{mat}} + \rho_{\rm ext} \tag{4.10}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{4.11}$$

or

$$\nabla \cdot \boldsymbol{D} = \rho_{ext} \tag{4.12}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{4.13}$$

where the *electric displacement* is

$$\boldsymbol{D} \equiv \boldsymbol{E} + \boldsymbol{P} \tag{4.14}$$

(e) For a linear isotropic medium

$$\boldsymbol{D} = (1+\chi)\boldsymbol{E} \equiv \varepsilon \boldsymbol{E} \tag{4.15}$$

but in general $\boldsymbol{D}$ is a function of $\boldsymbol{E}$ which must be specified before problems can be solved.

Working problems with Dielectrics

(a) Using Eq. (4.9) and the Eq. (4.12) we find the boundary conditions that *normal* components of $\boldsymbol{D}$ jump across a surface if there is external charge, while the *parallel* components $\boldsymbol{E}$ are continuous

$$\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \sigma_{\rm ext} \qquad D_{2\perp} - D_{1\perp} = \sigma_{\rm ext} \tag{4.16}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \qquad E_{2\parallel} - E_{1\parallel} = 0 \tag{4.17}$$

Very often $\sigma_{\rm ext}$ will be absent and then $D_\perp$ will be continuous (but *not* $E_\perp$).

(b) A jump in the polarization induces bound surface charge at the jump.

$$-\boldsymbol{n} \cdot (\boldsymbol{P}_2 - \boldsymbol{P}_1) = \sigma_{\rm mat} \tag{4.18}$$

(c) Since the curl of $\boldsymbol{E}$ is zero we can always write

$$\boldsymbol{E} = -\nabla\varphi \tag{4.19}$$

and for linear media ($\boldsymbol{D}(\boldsymbol{r}) = \varepsilon(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})$) with a non-constant dielectric constant $\varepsilon(\boldsymbol{r})$, we find an equation for $\boldsymbol{D}$

$$\nabla \cdot \varepsilon(\boldsymbol{r})\nabla\varphi = 0 \tag{4.20}$$

(d) With the assumption of a linear medium $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and constant dielectric constant, the equations for electrostatics in medium are essentially identical to electrostatics without medium

$$-\varepsilon\nabla^2\Phi = \rho_{ext} \,, \tag{4.21}$$

but, the new boundary conditions lead to some (pretty minor) differences in the way the problems are solved.

Energy and Stress in Dielectrics:

(a) We worked out the extra energy stored in a dielectric as an ensemble of external charges are placed into the dielectric. As the macroscopic electric field $\boldsymbol{E}$ and displacement $\boldsymbol{D}(\boldsymbol{E})$ are changed by adding external charge $\delta\rho_{ext}$, the change in energy stored in the capacitor material is

$$\delta U = \int_V {\rm d}^3x \, \boldsymbol{E} \cdot \delta\boldsymbol{D} \tag{4.22}$$

(b) For a linear dielectric δU can be integrated, becoming

$$U = \tfrac{1}{2}\int_V {\rm d}^3x \, \boldsymbol{E} \cdot \boldsymbol{D} = \tfrac{1}{2}\int_V {\rm d}^3x \, \varepsilon\boldsymbol{E}^2 \tag{4.23}$$

(c) We worked out the stress tensor for a linear dielectric and found

$$T_E^{ij} = -\tfrac{1}{2}(D^iE^j + E^iD^j) + \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}\delta^{ij} \tag{4.24}$$

$$= \varepsilon\left(-E^iE^j + \frac{1}{2}\boldsymbol{E}^2\delta^{ij}\right) \tag{4.25}$$

where in the first line we have written the stress in a form that can generalize to the non-linear case, and in the second line we used the linearity to write it in a form which is proportional the vacuum stress tensor.

(d) As always the force per volume in the Dielectric is

$$f^j = -\partial_i T_E^{ij} \tag{4.26}$$

where

$$T^{ij} = \text{the force in the } j\text{-th direction per area in the } i\text{-th} \tag{4.27}$$

More precisely let $\boldsymbol{n}$ be the (outward directed) normal pointing from region LEFT to region RIGHT, then

$$n_i T^{ij} = \text{the } j\text{-th component of the force per area, } \textit{by region LEFT on region RIGHT} \quad . \tag{4.28}$$

We can integrate the force/volume to find the net force on a given volume

$$F^j = \int_V d^3x \, f^j(\boldsymbol{x}) = -\int_{\partial V} da_i \, T^{ij} \tag{4.29}$$

This can be used to work out the force at a dielectric interface as done in lecture.