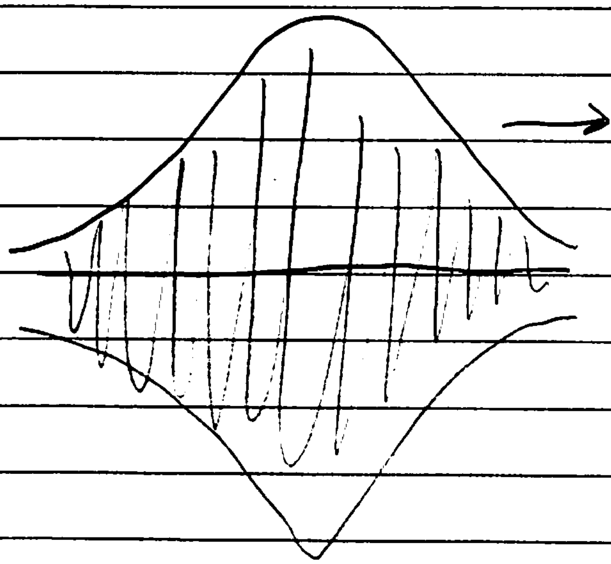
Wave Packets

- So far we have been considering individual plane waves. A general wave is a superposition of plane waves



The wave packet should also be a solution to the Helmholtz equations. This means for every $\vec{k}$, there is an $\omega(\vec{k})$. We will assume $\omega(\vec{k})$ real. In general there is imaginary part. Then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega(k)t} \approx \sum_k A_k e^{ik_n x - i\omega_k t}$$

The shape of the initial packet determines $A(k)$

$$u(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx} \Longrightarrow A(k) = \int_{-\infty}^{\infty} dx\, u(x,0)\, e^{-ik\cdot x}$$

Wave Packets Pg. 2

- Recall some fourier transforms

Gaussion: $G(x) \equiv C e^{-x^2/4\sigma^2} \longleftrightarrow \hat{G}(k) = \tilde{C} e^{-k^2\sigma^2}$

phase: $e^{ik_0x} f(x) \longleftrightarrow \hat{f}(k-k_0)$

vs

shift $f(x-x_0) \longleftrightarrow e^{-ikx_0}\hat{f}(k)$

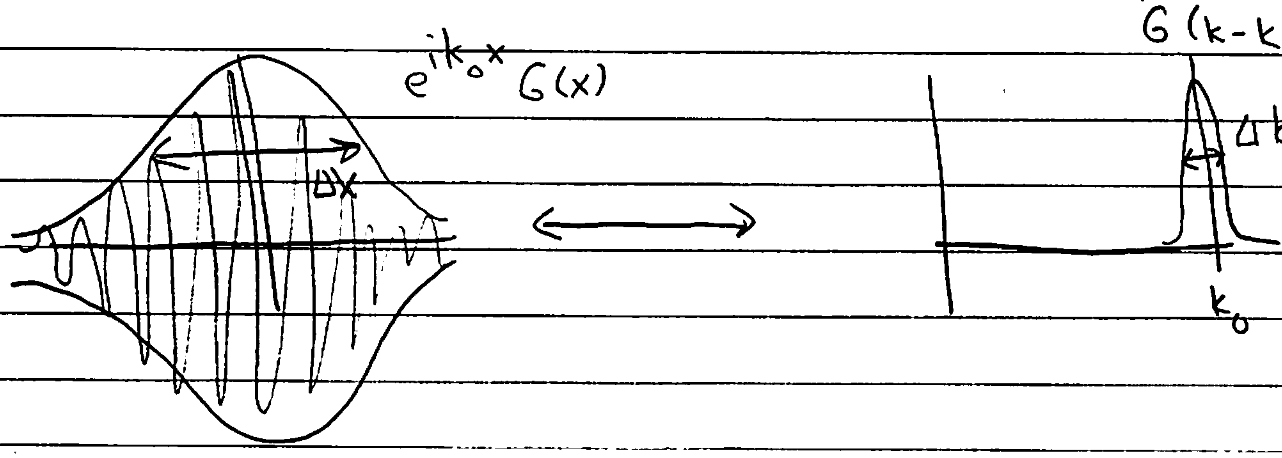
- The uncertainty principle applies to

$$(\Delta X)^2 = \int dx\, |u(x,0)|^2 \,(x-\bar{x})^2$$

$$(\Delta k)^2 = \int \frac{dk}{2\pi} |A(k)|^2 (k-\bar{k})^2$$

find, $\Delta k\, \Delta X \geq \frac{1}{2}$ with equality holding (uniquely) for gaussian

- So a typical wave packet



$\Delta x \sim \frac{1}{\Delta k}$

where $\Delta k \ll k_0$ since $k_0 \Delta x \sim \frac{k_0}{\Delta k} \gg 1$

- Then, lets ask about the solution at future times:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{+ikx - i\omega(k)t}$$

And we expand near k_0

$$\omega(k) \simeq \omega(k_0) + \underbrace{\frac{d\omega_0}{dk}}_{d\omega/dk|_{k=k_0}} (k-k_0)$$

So

$$u(x,t) = \underbrace{e^{+i[d\omega_0/dk\, k_0 - i\omega(k_0)]t}}_{e^{i\phi_0 t}} \int_{-\infty}^{\infty} dk/2\pi\, e^{ikx - d\omega_0/dk\, kt} A(k)$$

$$= e^{i\phi_0 t} \int_{-\infty}^{\infty} e^{ik(x - d\omega_0/dk\, t)} A(k)$$

$$u(x,t) = e^{i\phi_0 t}\, u(x - d\omega_0/dk\, t)$$

Thus we see that apart from an irrelevant phase, the wave packet travels with a speed given by

$$\boxed{v_g = \left.\frac{d\omega}{dk}\right|_{k_0}}$$

group velocity

For

$$\omega(k) = \frac{ck}{n(k)}$$

$$\frac{d\omega}{dk} = \frac{c}{n(k)} - \frac{ck}{n^2}\frac{dn}{d\omega}\frac{d\omega}{dk}$$

Solve

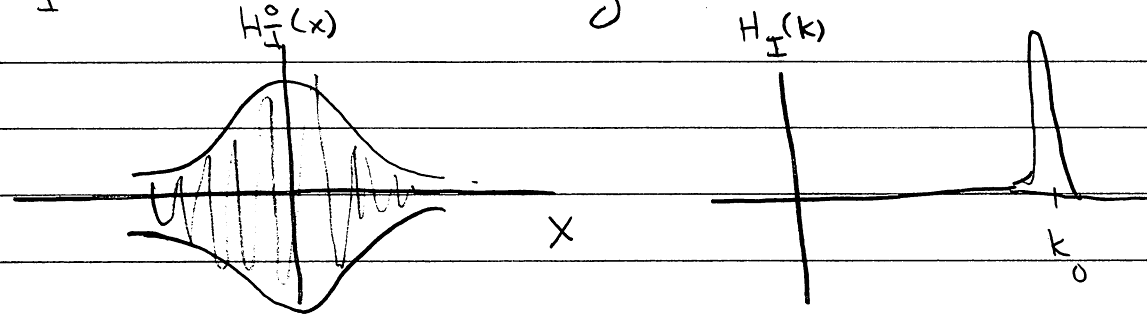
$$\frac{d\omega}{dk} = \frac{c}{n(\omega) + dn/d\omega}$$

Comments on Homework

The setup is the following. A wave-packet has the following form

$$H_I^0(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, H_I(k) \, e^{ikx} \quad \leftarrow \text{represents the magnetic field}$$

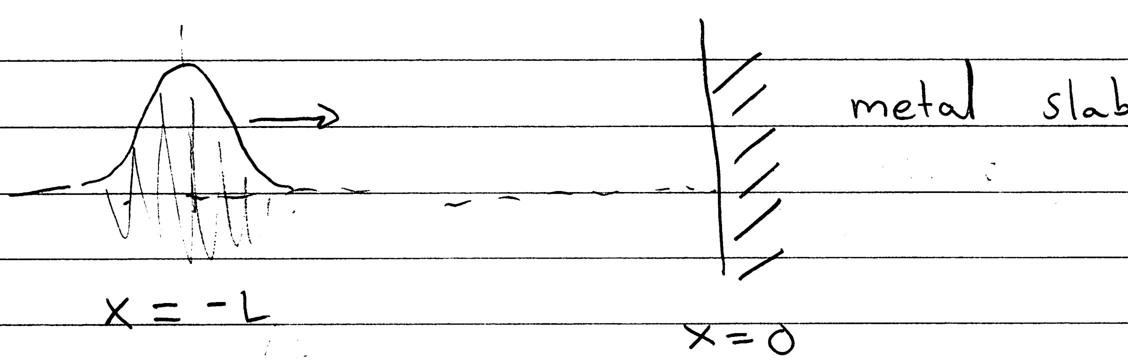
where $H_I^0(x)$ looks something like this:



Now consider an incoming wave-packet

$$H_I(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, H_I(k) \, e^{ikx - ckt} = H_I^0(x - ct)$$

Thus the wave packet at time $t = -\frac{L}{c}$



is centerred at $x = -L$. This is where $x - ct$ is zero

The reflected wave as a fcn of x and t is:

$$H_R(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, H_R(k)\, e^{-ikx - ickt}$$

We previously showed that

$$\frac{H_R(k)}{H_I(k)} \simeq 1 - \sqrt{\frac{2\mu\omega}{\sigma}}\,(1-i) \equiv r(k)\, e^{i\phi(k)}$$

where $r(k) \simeq 1 - \sqrt{2\mu\omega/\sigma}$ and $\phi(k) \simeq \sqrt{2\mu\omega/\sigma}$

Show that the center of the wave packet returns to $x = -L/c$ at time

$$t = \frac{L}{c} + \frac{\mu\delta_0}{2c}$$

with $\delta_0 = \sqrt{2c/\sigma\mu k_0}$ is the skin depth evaluated at the central (or mean) wave number of the packet.