Electrostatics Jackson 1.7

Fundamental Equations are

$$\nabla \cdot E = \rho \qquad \text{and} \qquad \int \vec{E} \cdot d\vec{S} = Q_{enc}$$

$$\nabla \times E = 0 \qquad \text{and} \qquad \oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{F} = q\vec{E}$$

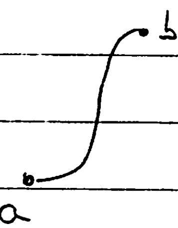
The main objective here is to compute the forces and interaction energies of charged objects and to learn math.

Since $\nabla \times \vec{E} = 0$ it can be written as the gradient of a scalar function (Helmholtz theorem)

$$\star \qquad E = -\nabla \varphi \leftarrow \varphi \text{ is the scalar potential (voltage)}$$

Alternaltively

$$\varphi(\vec{x}_b) - \varphi(\vec{x}_a) = -\int_{\vec{x}_a}^{\vec{x}_b} \vec{E} \cdot d\vec{l}$$



Substituting $\star$ into $\nabla \cdot E = \rho(x)$ we find an equation for φ, using $\nabla \cdot (\vec{\nabla} \varphi) = \nabla^2 \varphi$

$$\boxed{-\nabla^2 \varphi = \rho} \leftarrow \text{poisson equation}$$

When $\rho = 0$

$$-\nabla^2 \varphi = 0 \quad \text{Laplace equation}$$

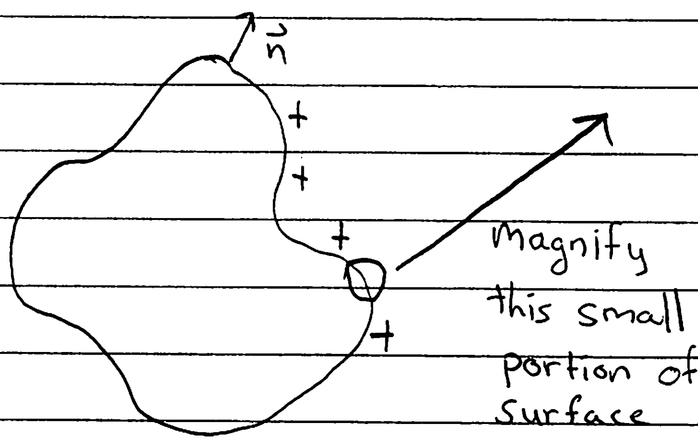
Technical note:

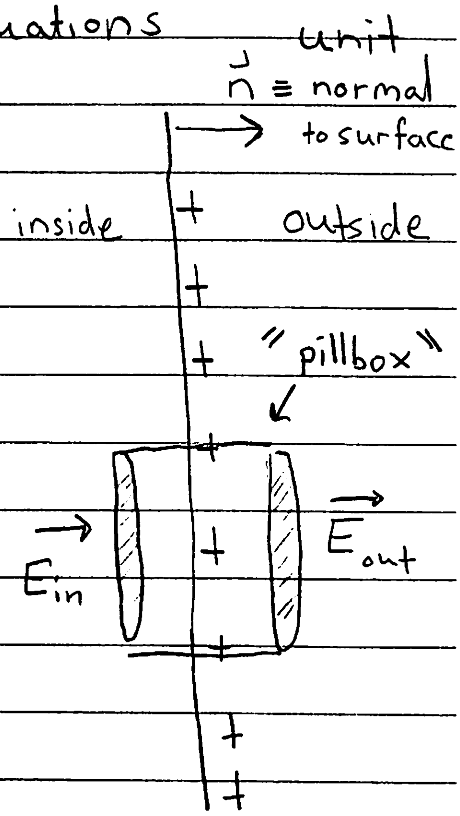
- I will generally place a minus sign in front of ∇^2. The reason for this is because $-\nabla^2$ is a positive semi-definite linear operator. For any function $\Psi(x)$

$$\int d^3x \; \Psi^*(x) \left(-\nabla^2 \Psi\right) \geq 0$$

Boundary Conditions. Jackson 1.6

The boundary conditions across a surface are obtained by integrating the equations across the surface:





Sufficiently close to the surface it looks like a plane of charge. Applying Gaus Law to the "pillbox" we have

$$\oint \vec{E}\cdot d\vec{a} = Q_{enc}$$

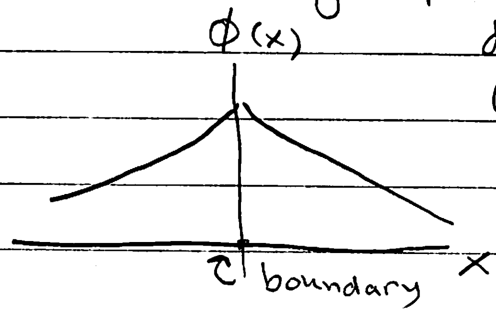
$$A\,(\vec{n}\cdot\vec{E}_{out} - \vec{n}\cdot\vec{E}_{in}) = Q_{enc}\,, \qquad \text{or}$$

area of face of pillbox

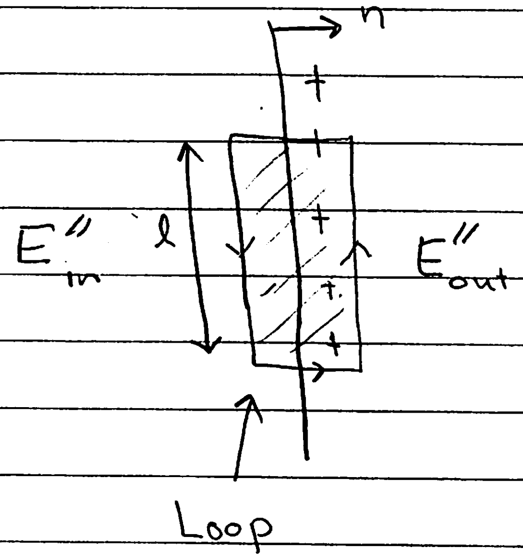
$$\boxed{n\cdot\vec{E}_{out} - \vec{n}\cdot\vec{E}_{in} = \sigma}$$

where $\sigma = Q/A$ charge per area on surface. The derivative of the potential (the E-field) jumps across the surface in proportion to σ.

Picture:



Similarly: we integrate over the surface (see figure)



$$\int d\vec{S} \cdot \nabla \times E = 0$$

So

$$\oint_{\text{Loop}} \vec{E} \cdot d\vec{l} = 0$$

that is neglecting top and bottoms of Loop:

$$l\,(E^{\parallel}_{out} - E^{\parallel}_{in}) = 0$$

$$\boxed{E^{\parallel}_{out} - E^{\parallel}_{in} = 0}$$

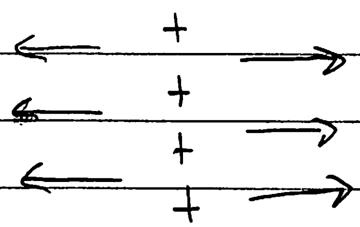
Here in $E^{\parallel}_{out/in}$ are the components of $\vec{E}$ that are parallel to the surface. In general there two such components. We can write it as a vector equation

$$\boxed{\vec{n} \times (\vec{E}_{out} - \vec{E}_{in}) = 0}$$

Boundary Conditions and Forces on a Charged Plate

First recall that for a charged plane, the electric field is $\frac{\sigma}{2}$ by Gauss Law

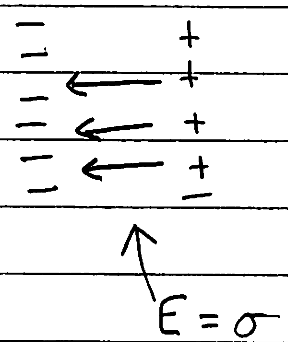
$E_{Left} = \frac{-\sigma}{2}$



$E_{right} = \frac{\sigma}{2}$

This satisfies our boundary conditions, $E_{right} - E_{left} = \sigma$.
Then a for a capacitor, we get:

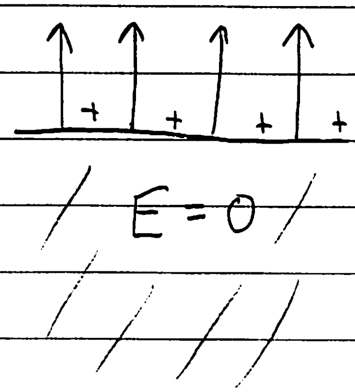
$E = 0$



$E = 0$

For metal block, the electric field is normal to the surface (since $E^{\parallel} = 0$ inside the metal). Outside the metal,

E_{out}

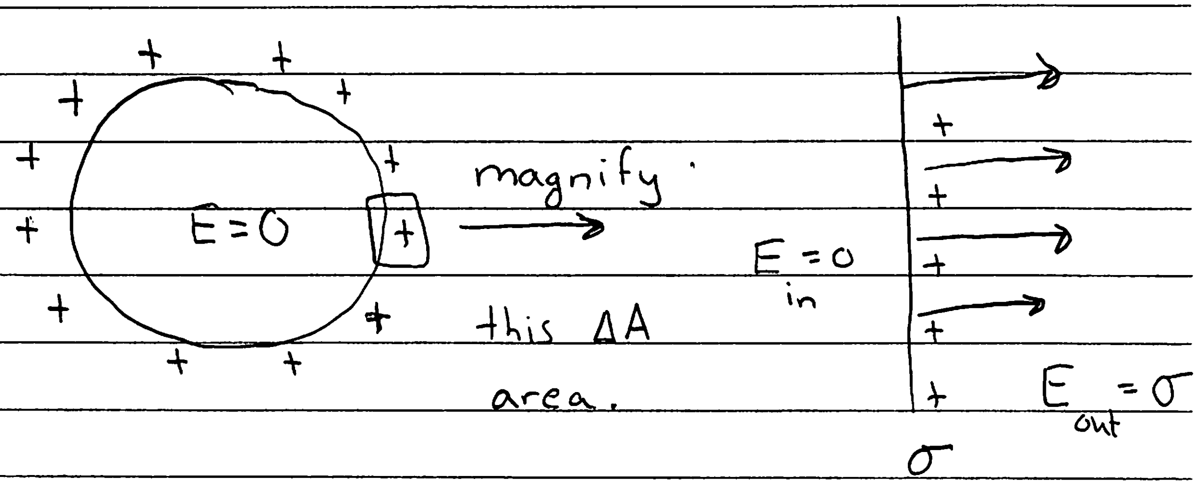


$\vec{E}_{out} = E_n \hat{n} \qquad E_n = \sigma,$

$\vec{E}_{in} = 0$

So $E_{out} - E_{in} = \sigma$ as required.

Now consider a charged solid metal object and determine the force per area on the surface:



Ask about the force per area on this surface ΔA:

$$\frac{F}{\Delta A} = \sigma \left(E_{out} - E_{self} \right)$$

Charge per area

the part of the electric field produced by σ. We do not want to include this self force

The electric field produced by a wall of charge is $E_{self} = \sigma/2$ (see previous page) while the full electric field outside the metal is $E_{out} = \sigma$. So

$$\frac{F}{\Delta A} = \sigma \left(\sigma - \frac{\sigma}{2} \right)$$

$$\boxed{\frac{F}{\Delta A} = \frac{\sigma^2}{2}}$$

We will derive this again using the stress tensor