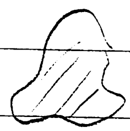
How to solve the Laplace + Poisson Equation - Green Fcn Intro

$\rho(\vec{r})$



Want to solve

$$-\nabla^2 \varphi(r) = \rho(r)$$

First limit the disccussion to free space, $\Phi \xrightarrow[r\to\infty]{} 0$.
Then the potential is given by

$$\varphi(\vec{r}) = \int_{r_0} \frac{\rho(\vec{r}_0)}{4\pi|\vec{r}-\vec{r}_0|}$$

Formally we have a green function, which is a function which satisfies the diff-eq with a δ-fcn source

$$-\nabla_r^2 G(\vec{r},\vec{r}_0) = \delta^3(\vec{r}-r_0)$$

Then the solution is the convolution of the Green function and the source, $\rho(r_0)$:

$$\varphi(\vec{r}) = \int d^3r_0 \, G(\vec{r},\vec{r}_0)\,\rho(r_0)$$

Then

$$-\nabla_r^2 \varphi(\vec{r}) = \int d^3r_0 \, \underbrace{-\nabla^2 G(r,r_0)}_{\delta^3(\vec{r}-r_0)} \rho(r_0)$$

Or

$$-\nabla_r^2 \varphi(r) = \rho(r)$$

More physically, the Green-fcn $G(\vec{r}, \vec{r}_0)$ is the potential at $\vec{r}$, due to a unit point charge at $\vec{r}_0$

For free space we know

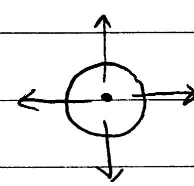
$$G_0(\vec{r}, \vec{r}) = \frac{1}{4\pi |\vec{r} - \vec{r}_0|}$$ (the free space Green function G_0)

Check that $-\nabla^2 G_0 = \delta^3(r, r_0)$, by first noting that

$$-\nabla^2 \frac{1}{4\pi r} = 0 \qquad \text{except at } r = 0$$

Then can verify (using Gaus Law) that

$$\int_{\substack{\text{ball}\\ \text{around } \vec{0}}} dV \; -\nabla^2 \frac{1}{4\pi r} = 1$$

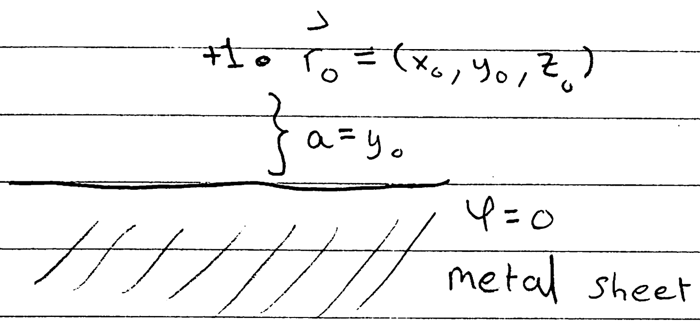


This you show by using $-\nabla^2(1/4\pi r) = \vec{\nabla} \cdot \hat{r}/4\pi r^2$, so

$$\int dV \, \vec{\nabla} \cdot \hat{r}/4\pi r^2 = \int \underbrace{r^2 d\Omega \, \hat{r}}_{\equiv d\vec{a}} \cdot \frac{\hat{r}}{4\pi r^2} = 1$$

Solving for Dirchlet Green Fcn with images

$\varphi(\vec{r}) = 0$

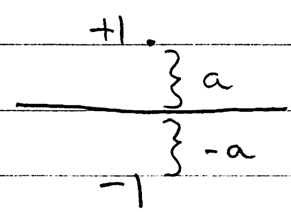


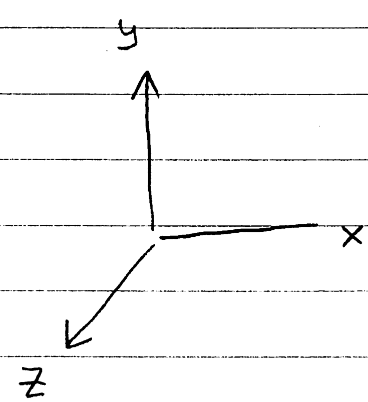
The Green fcn is the potential at $\vec{r}$, due to a point charge at $\vec{r}_0$. Formally want to solve

$$-\nabla^2 G_D(\vec{r}, \vec{r}_0) = \delta^3(\vec{r} - \vec{r}_0) \qquad \text{for} \quad y > 0$$

with a boundary condition that $\varphi(\vec{r})$ vanishes as $y \to 0$ (the boundary metal sheet). Such a Green function with vanishing boundary conditions is known as a Dirichlet Green Function.

Solution: place an image charge at $y_I = -a$ with opposite sign





The potential is

$$G(r, \vec{r}_o) = \underbrace{\frac{1}{4\pi |\vec{r} - \vec{r}_o|}}_{\text{this is } G_o} + \underbrace{\frac{-1}{4\pi |\vec{r} - \vec{r}_{oI}|}}_{\text{this is } G_{induced}}$$

where $\vec{r}_{oI} = (x_o, -y_o, z_o)$ is the location of the image charge. The solution takes the form:

$$\boxed{G_D(\vec{r}, \vec{r}_o) = \underbrace{G_o(\vec{r}, \vec{r}_o)}_{\substack{\text{the singular free} \\ \text{coulomb Grn fcn}}} + \underbrace{G_{ind}(\vec{r}, \vec{r}_o)}_{\substack{\text{The regular induced} \\ \text{potential by boundaries}}}}$$

Where $G_{ind}(\vec{r})$ obeys the homogeneous equation, $-\nabla^2 G_{ind} = 0$, and G_o is the free coulomb expresion. Note G_{ind} is regular in the upper half plane (our problem domain).

The interaction energy between a charge q is:

U_{int} = interaction between plane and charge q at $\vec{r}_o$

$$= q\,\varphi_{ind}(r)$$

$\varphi(r) = q\, G(\vec{r}, \vec{r}_o)$

(G is for a unit charge)

$$\boxed{U_{int} = q^2 \left[G(\vec{r}, \vec{r}_o) - G_o(\vec{r}, \vec{r}_o) \right]}$$

$$= -\frac{q^2}{4\pi |\vec{r} - \vec{r}_{oI}|} \quad \text{(in this case)}$$

So the force is:

$$F = -\nabla U_{int} = -\nabla \varphi_{ind}(\vec{r})$$

$$\vec{F} = -q^2 \frac{(\vec{r} - \vec{r}_{oI})}{4\pi |\vec{r} - \vec{r}_{oI}|^3}$$

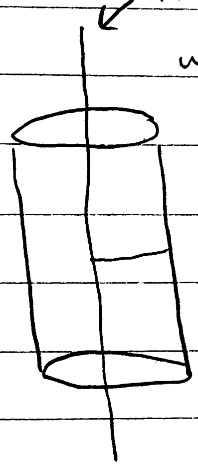
Green Fcn in 2D

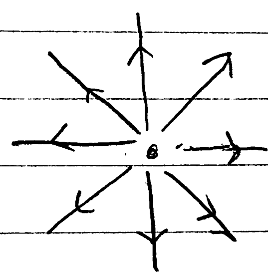
- The green function $G(\vec{r}, \vec{r}_0)$ is the potential at $\vec{r}$ due to a "point" charge at $\vec{r}_0$

In 2D, $\vec{r} = (x, y)$ and $\vec{r}_0 = (x_0, y_0)$, and G obeys:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(\vec{r}, \vec{r}_0) = \delta^2(\vec{r} - \vec{r}_0).$$

So $G(\vec{r}, \vec{r}_0)$ is the potential due to a line of chage in three dimensions





Use gauss law to find:

$$\varphi(\vec{r}) = \frac{-\lambda}{2\pi} \log |\vec{r}|$$

Thus the Green function (in 2D and free space) is

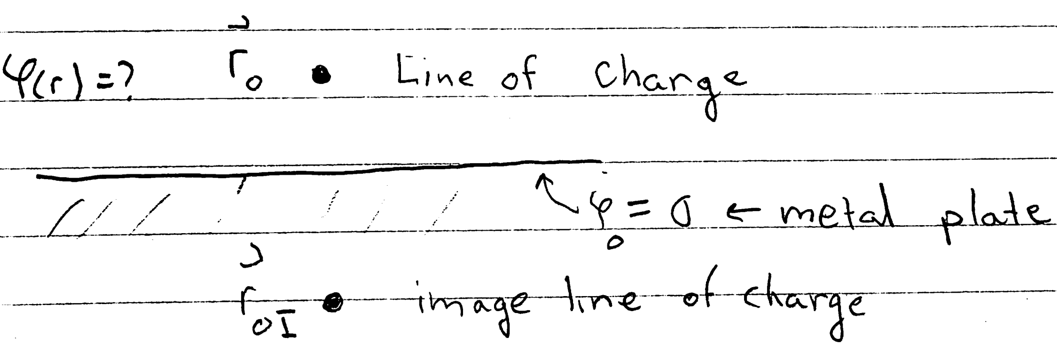
$$G_0(\vec{r}, \vec{r}_0) = \frac{-1}{2\pi} \log |\vec{r} - \vec{r}_0|$$

Then the potential due to a charge distribution is just a superposition (2D and free space)

$$\varphi(\vec{r}) = \int d^2\vec{r}_0 \, \rho(\vec{r}_0) \, G_0(\vec{r}, r_0)$$

$$= \int d^2 r_0 \, \rho(r_0) \left[-\frac{1}{2\pi} \log |\vec{r} - \vec{r}_0| \right]$$

We can also find the Green function for more complex geometries



$$G_D(\vec{r}, \vec{r}_0) = -\frac{1}{2\pi} \log |\vec{r} - \vec{r}_0| + \frac{1}{2\pi} \log |\vec{r} - \vec{r}_{0I}|$$

Dirichlet Green fcn, i.e. this is the Green function which vanishes on the boundary. $G_D(\vec{r}, \vec{r}_0)$ satisfies

$$-\nabla^2 G_D = \delta^2(\vec{r} - \vec{r}_0)$$

and vanishes on the boundary metal surface.