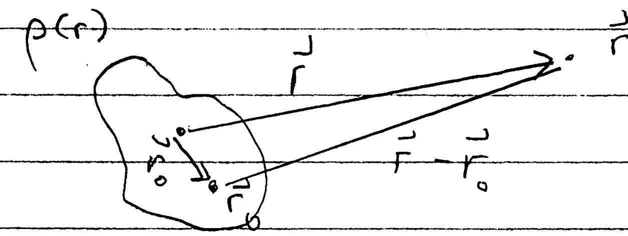
Multipole Expansion with Spherical Harmonics



Let us redo the multipole expansion

Then

$$\varphi(\vec{r}) = \int d^3\vec{r}_0 \, \frac{\rho(r_0)}{4\pi\,|\vec{r}-\vec{r}_0|}$$

For $r \gg r_0$ we can expand $r_> = r$ and $r_< = r_0$ and we have the expansion

$$\frac{1}{4\pi\,|\vec{r}-\vec{r}_0|} = \sum_{lm} \frac{r_0^l}{r^{l+1}} \frac{1}{(2l+1)} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta_0\phi_0)$$

This leads to

$$\varphi(r) = \sum_{lm} \frac{q_{lm}}{(2l+1)} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} = \frac{q_{00}Y_{00}}{r} + \frac{1}{3}\frac{q_{1m}Y_{1m}}{r^2} + O\left(\frac{1}{r^3}\right) \cdots$$

where

$$q_{lm} = \int d^3r_0 \, \rho(\vec{r}_0)\, r_0^l\, Y^*_{lm}(\theta_0,\phi_0)$$

spherical multipole moment

This multipole expansion is entirely equivalent to the expansion we had previously

$$\varphi(r) = \frac{Q_{TOT}}{4\pi r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \frac{Q_{ij}(\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij})}{4\pi r^3}$$

$$+ \ldots$$

To see this one needs to understand what $Y_{lm}(\theta,\phi)$ are. Y_{lm} are linearly combos of the components of a symmetric traceless l-th rank tensor constructed out of $\hat{r}$

$\hat{\boldsymbol{r}} \equiv \frac{\boldsymbol{r}}{r}$

Cartesian	Spherical	Rank
1	Y_{00}	0
$\hat{r}_i$	Y_{1m}	1
$\hat{r}_i\hat{r}_j - \frac{1}{3}\delta^{ij}$	Y_{2m}	2
$\hat{r}^i\hat{r}^j\hat{r}^k - \frac{1}{5}(\hat{r}^i\delta^{jk} + \hat{r}^j\delta^{ki} + \hat{r}^k\delta^{ij})$	Y_{3m}	3

And so on.

1 Spherical Harmonics Table

For $\ell = 0$ $r^0 Y_{00}$ is a linear combination of the scalar 1

$$Y_{00} \propto 1$$

For $\ell = 1$ rY_{1m} is a linear combination of the vector $r^i = (x, y, z)$

$$r^i \propto \begin{cases} x \\ y \\ z \end{cases} \qquad\qquad r^2 Y_{2m} \propto \begin{cases} (x+iy) \\ z \\ (x-iy) \end{cases}$$

For $\ell = 2$ $r^2 Y_{2m}$ is a linear combo of the (symm-traceless) 2nd rank tensor $3r^i r^j - r^2 \delta^{ij}$

$$3r^i r^j - r^2\delta^{ij} \propto \begin{cases} 3z^2 - r^2 \\ zx \\ zy \\ (x^2 - y^2) \\ xy \end{cases} \qquad\qquad r^2 Y_{2m} \propto \begin{cases} (x+iy)^2 \\ z(x+iy) \\ 3z^2 - r^2 \\ z(x-iy) \\ (x-iy)^2 \end{cases}$$

where $x^2 - y^2$ is $\propto$ the difference between the xx and yy components of $3r^i r^j - r^2\delta^{ij}$.

For $\ell = 3$

Y_{3m} is a linear combo of the (sym-traceless) tensor $\underbrace{5r^i r^j r^k - r^2(\delta^{ij} r^k + \delta^{jk} r^i + \delta^{ki} r^j)}_{\text{has 7-components for the seven } Y_{3m}}$

The transformation between cartesian and spherical is such that the dot products aagree

$$\boldsymbol{p} \cdot \hat{\boldsymbol{r}} = \frac{4\pi}{2\ell + 1} \sum_{m=-1}^{1} q_{1m} Y_{1m} \tag{1}$$

$$\tfrac{1}{2} \mathcal{Q}_{ij} (\hat{r}^i \hat{r}^j - \tfrac{1}{3}\delta^{ij}) = \frac{4\pi}{2\ell + 1} \sum_{m=-2}^{2} q_{2m} Y_{2m} \tag{2}$$

To understand my meaning, take the dipole term:

This notation is bad. I mean: $Y_{11} \propto \frac{(x+iy)}{r}$, $Y_{1,-1} \propto \frac{(x-iy)}{r}$, and $Y_{10} \propto \frac{z}{r}$

$$\left.\begin{aligned} Y_{11} &\propto (\hat{x} + i\hat{y}) \\ Y_{1,-1} &\propto (\hat{x} - i\hat{y}) \\ Y_{10} &\propto \hat{z} \end{aligned}\right\}$$

We see that Y_{1m} is a linear combo of $\hat{r}^i$

Similarly q_{1m} is a linear combo of $\vec{p}$, e.g.:

$$q_{11} = \int d^3 r_0 \, \underbrace{r_0 Y^*_{1m} \rho(r_0)}_{\propto (x-iy)\rho(r_0)} \propto \underbrace{p^x - i p^y}_{\text{The } x, y \text{ components of } \vec{p}}$$

since

$$p^x \equiv \int d^3 \vec{r}_0 \, x \, \rho(\vec{r}_0) \quad \text{etc.}$$

The relation between p^i and q_{1m} is the same as the relation (i.e. linear-combo) between $\hat{r}^i$ and Y^*_{1m}

The relations and normalizations are chosen so that the series agree, e.g.

$$\frac{\vec{p} \cdot \hat{r}}{4\pi r^2} = \sum_m \underbrace{\frac{1}{3}}_{2\ell+1 \text{ for } \ell = 1} \frac{q_{1m}}{r^2} Y_{1m} \Rightarrow \vec{p} \cdot \hat{r} = \frac{4\pi}{3} \sum_m q_{1m} Y_{1m}$$

This is the statement that

$$\vec{p}\cdot\hat{r} = \frac{(p_x - ip_y)}{\sqrt{2}}\frac{(\hat{r}_x + i\hat{r}_y)}{\sqrt{2}} + \frac{(p_x + ip_y)}{\sqrt{2}}\frac{(\hat{r}_x - i\hat{r}_y)}{\sqrt{2}} + p_z \cdot \hat{r}_z$$

$$= \frac{4\pi}{3}\left(q^*_{11} Y_{11} + q^*_{1-1} Y_{1-1} + q_{10} Y_{10}\right)$$

Similarly Y_{2m} is a linear combo of $\hat{r}^i\hat{r}^j - \frac{1}{3}\delta^{ij}$

(There are five components of $\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij}$, and five $\ell = 2$ spherical harmonics). And, q_{2m} is a linear combo of the quadrupole tensor Q_{ij} components (The map between q_{2m} and Q_{ij} is the same as between Y^*_{2m} and $\hat{r}^i\hat{r}^j - 1/3\,\delta^{ij}$). Then this map is constructed so that

$$\frac{1}{4\pi r^3} Q_{ij}\left(\hat{r}^i\hat{r}^j - \frac{1}{3}\delta^{ij}\right) = \sum_m \frac{1}{5} q_{2m} Y_{2m}/r^3$$

$\leftarrow$ $2\ell+1$ with $\ell=2$

Relation between the cartesian and spherical multipoles, $q, \boldsymbol{p}, Q_{ij} \leftrightarrow q_{00}, q_{1m}, q_{2m}$.

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\mathbf{x}')\, d^3x' = \frac{1}{\sqrt{4\pi}}\, q \tag{4.4}$$

$$\left.\begin{aligned} q_{11} &= -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\rho(\mathbf{x}')\, d^3x' = -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y) \\ q_{10} &= \sqrt{\frac{3}{4\pi}} \int z'\rho(\mathbf{x}')\, d^3x' = \sqrt{\frac{3}{4\pi}}\, p_z \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} q_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x' - iy')^2\rho(\mathbf{x}')\, d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{11} - 2iQ_{12} - Q_{22}) \\ q_{21} &= -\sqrt{\frac{15}{8\pi}} \int z'(x' - iy')\rho(\mathbf{x}')\, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} - iQ_{23}) \\ q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2)\rho(\mathbf{x}')\, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33} \end{aligned}\right\} \tag{4.6}$$