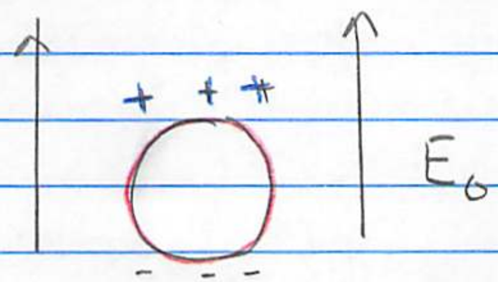
Separation of Variables in Spherical Coordinates

Consider a neutral metal sphere placed in an electric field $\vec{E} = E_0 \hat{z}$. The sphere has radius a



Separation of variables

Boundary conditions:

① $\varphi(r,\theta)\Big|_{r=a} = \varphi_0 = \text{constant}$

② As $r \to \infty$ we should approach $\vec{E} = -\nabla\varphi = E_0\hat{z}$

$$\varphi(r,\theta) \xrightarrow[r\to\infty]{} -E_0 z = -E_0 r\cos\theta,$$

Up to a constant. In addition, we know that the potential from the spere falls faster than $1/r$ since the sphere is neutral. So

$$\varphi \xrightarrow[r\to\infty]{} -E_0 r\cos\theta + \text{const} + O(1/r^2).$$

Overview

- After separating variables we will see that

$$\varphi(r,\theta) = \sum_{\ell} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

with $\ell = 0, 1, 2, 3, \ldots$ and P_ℓ the legendre polynom.

- The A_ℓ and B_ℓ are then adjusted to match the b.c.

- You may wish to skip the next several pages to see this in action

Separating Variables. Eigenvalue Problem.

- In spherical coordinates (with no ϕ dependence)

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{-1}{r^2}\frac{\partial}{\partial \sin\theta} \sin\theta \frac{\partial}{\partial\theta} \right] \varphi = 0$$

Let $x \equiv \cos\theta$ so $d\Omega = \sin\theta \, d\theta \, d\phi = dx \, d\phi$.

- Also try separated solution

$$\varphi = R(r)\, \mathbb{P}(x)$$

- Then $-\frac{r^2}{\varphi}\nabla^2\varphi = 0$ yields with $\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial x}$

$$-\frac{1}{R}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \frac{-1}{\mathbb{P}}\frac{\partial}{\partial x}(1-x^2)\frac{\partial \mathbb{P}}{\partial x} = 0$$

Leading to

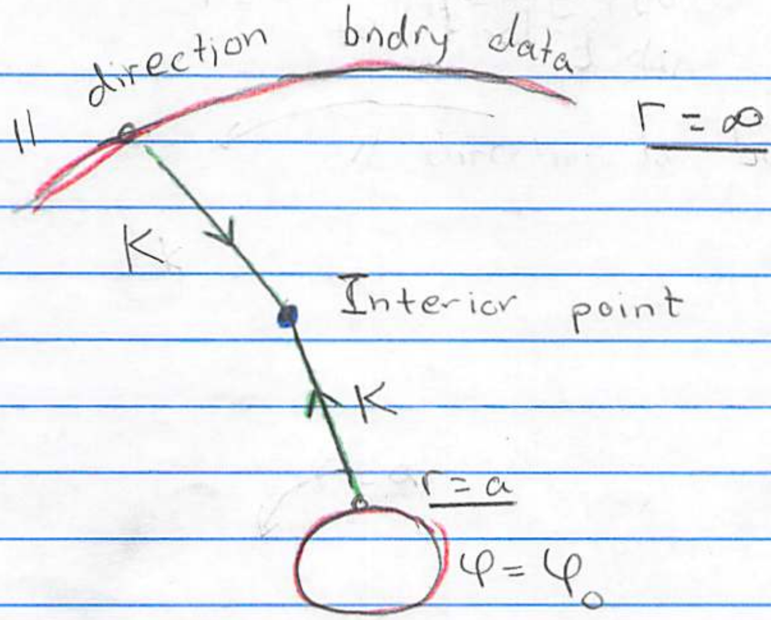
Separated Equations

$$-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_\nu + \frac{\nu(\nu+1)}{r^2} R_\nu = 0$$

$$-\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\mathbb{P}_\nu = \underbrace{\nu(\nu+1)}_{\text{constant}}\mathbb{P}_\nu$$

constant. later we will see that $\nu = \ell = 0, 1, 2, 3, \ldots$

- We expect that the direction // to the boundary data will lead to an e-value problem. While the perpendicular direction encodes the propagation



- The differential equation has singular points at $x=\pm 1$ and is a Sturm Liouville Equation

$$-\frac{\partial}{\partial x} p(x) \frac{\partial \mathbb{P}}{\partial x} = \nu(\nu+1)\mathbb{P} \qquad p(x)=(1-x^2)$$

- For $x=\pm 1$ the general solution very near $x=\pm 1$ is

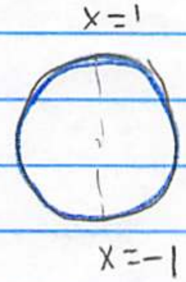
$$\mathbb{P} = C_1 + C_2 \log(1-x) \qquad \text{near } x=1$$

$$\mathbb{P} = D_1 + D_2 \log(1+x) \qquad \text{near } x=-1$$

- We wish our solutions to be regular at the north pole $(x=1)$ and the south pole $(x=-1)$. So we require $C_2=0$ and $D_2=0$

Thus we require regularity at $x=\pm 1$ as our b.c. This is a homogeneous b.c.

- We thus have homogeneous b.c. at either end of the interval. (Physically this is the requirement that the wave "fit" on the sphere, much the way requiring the wave "fits" in the box for particle in box quantum problem)



- Only for specific values of ν will we be able to satisfy these b.c. leading to an eigenvalue problem.

$$\mathcal{L}_x P_\ell(x) = \ell(\ell+1) P_\ell(x)$$

we use ν for any old value, and ℓ for itegers $\ell = 0, 1, 2, 3$ the eigenvalues

Where

$$\mathcal{L}_x = -\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}$$

- You can check that $\mathcal{L}_x$ is self adjoint with these b.c.:

$$\int_{-1}^{1} dx\, (\mathcal{L}_x f)^* g = (1-x^2)\left(f^* \frac{\partial g}{\partial x} - g\frac{\partial f^*}{\partial x}\right)\Bigg|_{-1}^{1}$$

$$+ \int_{-1}^{1} dx\, f^* (\mathcal{L} g)$$

i.e. the boundary term vanishes at $x=1$ and $x=1$ provided f and g are regular there

- The eigen-values turn out to be $\nu = \ell$ with $\ell = 0, 1, 2, 3 \ldots$

- The eigen-functions are the Legendre Polynomials

$$P_\ell(x) = \begin{cases} 1 & \ell = 0 \\ x & \ell = 1 \\ \frac{3}{2}x^2 - \frac{1}{2} & \ell = 2 \\ \vdots \text{ etc} & \end{cases}$$

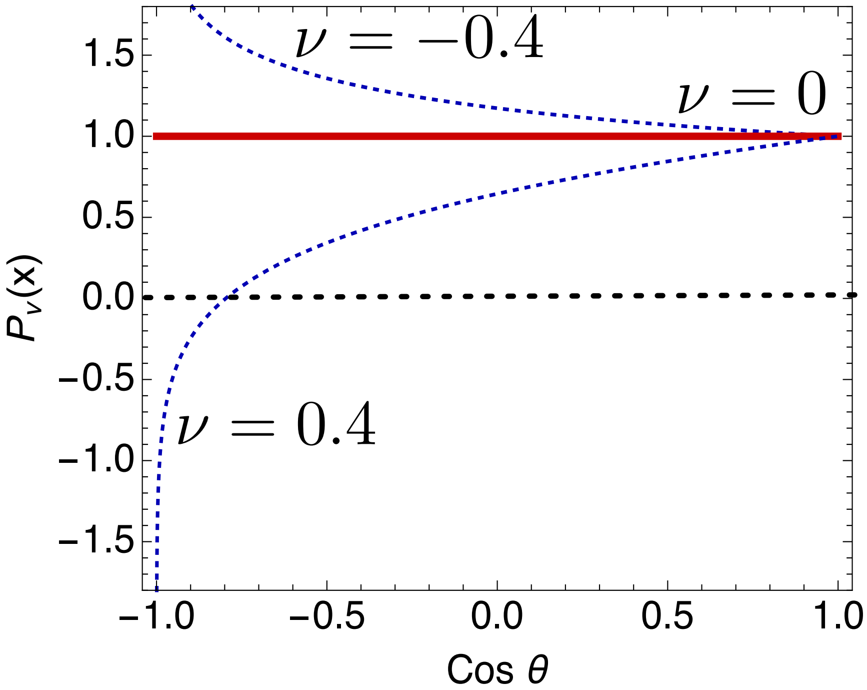
- To show this a numerical procedure consists of the following: ① Pick a ν, start at $x = 1$ where $P_\nu(x)$ has a series solution, which you find from the diffEQ:

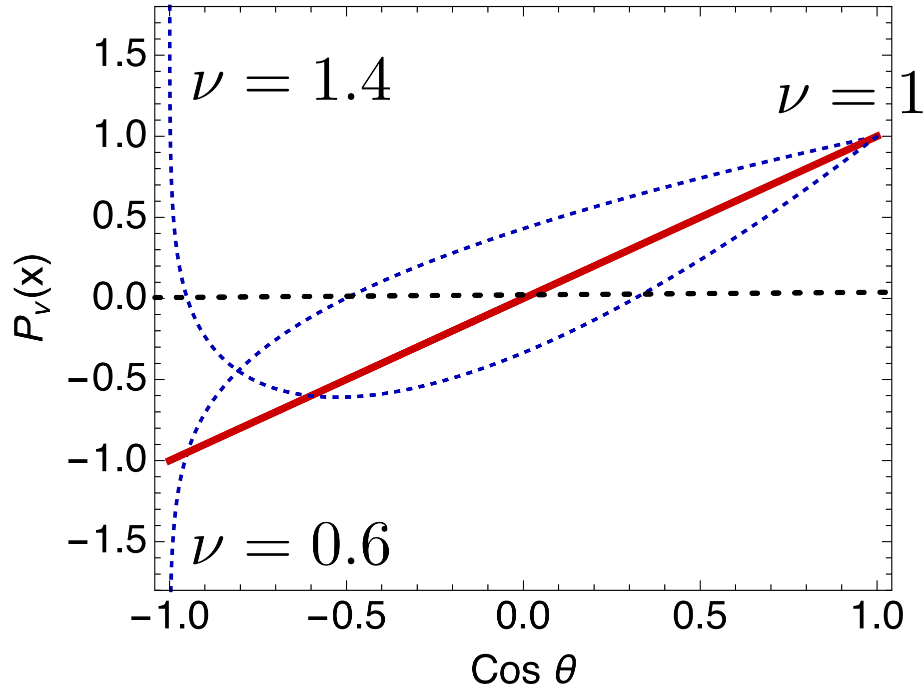
$$☆ \quad P_\nu(x) = 1 - \frac{\nu(\nu-1)}{2}(1-x) + O(1-x^2)$$

② Using ☆ as initial condition, integrate the diffEQ to $x = -1$. ③ Generally $P_\nu(x)$ will be irregular there and behave as $\sim \log(1+x)$. But if you choose ν to be an integer $P_\nu(x) \xrightarrow[x \to -1]{} \pm 1$

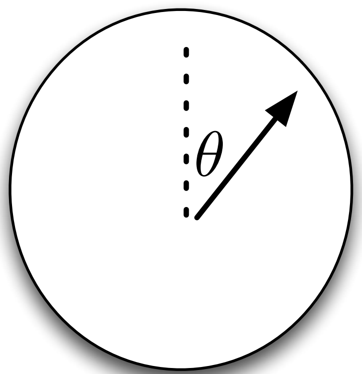
- Examining the handout, we see that $P_\nu(x)$ is at $x=-1$ regular when ν is an integer.

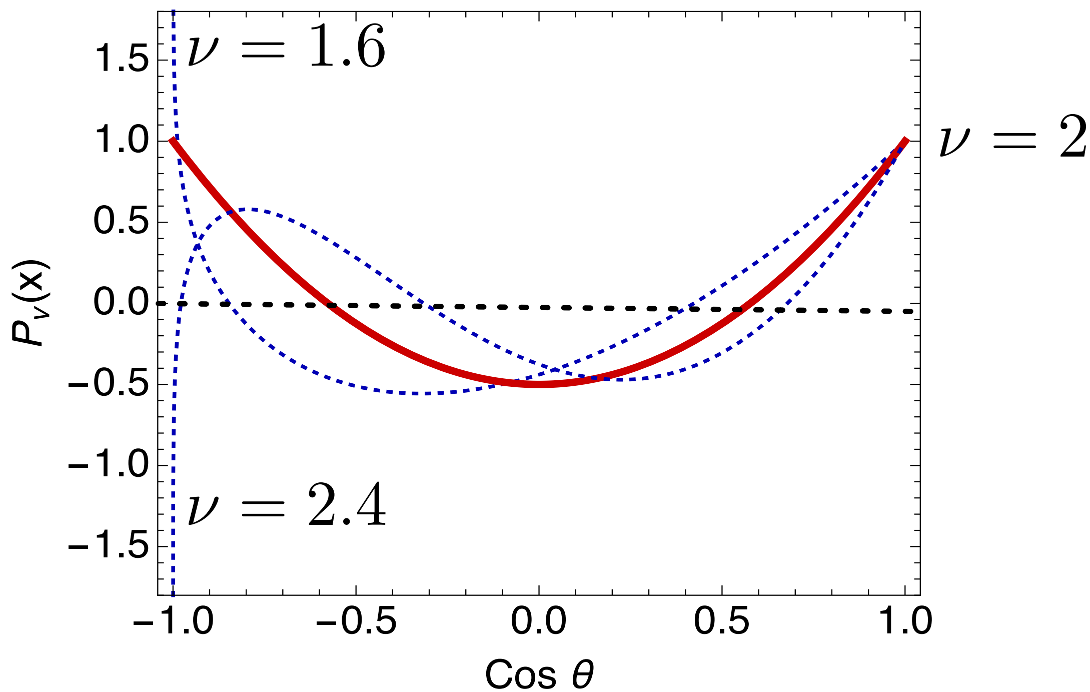
- An analytic approach to the e-value problem is discussed by jackson





Dial the slope at $\cos(\theta) = 1$ with ν





Now that ℓ is specified we return to the radial equation

$$\left[\frac{-1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] R = 0$$

Try a solution r^a and find $a = \ell, -(\ell+1)$ leading to

$$R_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

So the general form is

$$\varphi(r, \theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Solving the boundary problem

- The boundary data involve the $\ell = 0$ mode at $r = a$ (constant), and the $\ell = 1$ mode $\cos\theta$ as $r \to \infty$

- So it is reasonable to expect a solution involving only these modes

$$\varphi = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

- From $r \to \infty$, and $\varphi \to -E_0 r\cos\theta + \text{const} + \mathcal{O}\left(\frac{1}{r^2}\right)$ we find

$$A_1 = -E_0 \qquad B_0 = 0$$

- From $r = a$, $\varphi = 0$ boundary condition

$$A_0 = 0 \qquad A_1 a + \frac{B_1}{a^2} = 0$$

i.e.

$$B_1 = -E_0 a^3$$

So

$$\varphi = \left(-E_0 r + \frac{E_0 a^3}{r^2}\right)\cos\theta$$

Comparison of this with the multipole expansion

$$\varphi_{ind} = q_{TOT}/4\pi r + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \frac{Q_{ij}(\hat{r}^i\hat{r}^j - 1/3\,\delta_{ij})}{4\pi r^3} + \dots$$

induced potential

We find that $q_{TOT} = 0$ and

$$\boxed{\vec{p} = 4\pi E_0 a^3 \hat{z}}$$

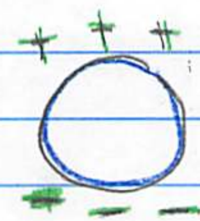
induced dipole moment

We can also compute the surface charge and show this directly

$$\vec{n}\cdot\vec{E}_{out} - \cancelto{0}{\vec{n}\cdot\vec{E}_{in}} = \sigma$$

$$-\frac{\partial\varphi}{\partial r}\Big|_{r=a} = \sigma$$

$\sigma = 3E\cos\Theta$



dipole moment $\quad 3E_0\cos\theta = \sigma$

$$p^z = \int a^2\, d\Omega\; \sigma(\theta)\; \overset{a\cos\theta}{z} = a^3\int_0^{2\pi} d\phi \int_{-1}^{1} dx\; 3E_0 x^2$$

$p^z = 4\pi E_0 a^3$ as above