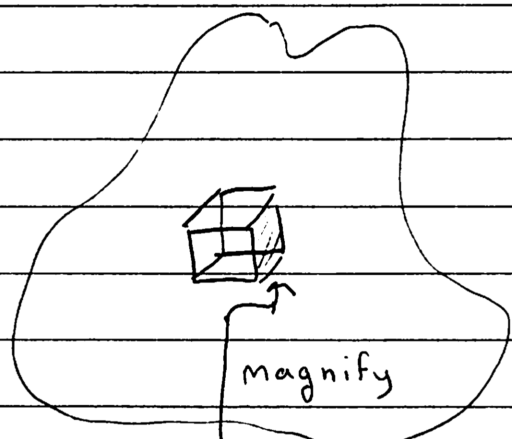
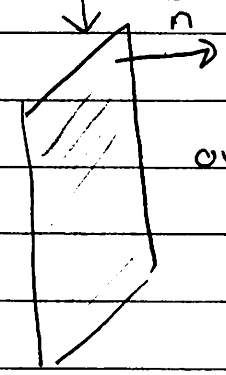
Force and Stress



When considering continuous macroscopic bodies (solids and fluids etc), we are interested in the force per volume. If we take a fluid cell, we can look at the forces on all of the faces of the cell, to determine the net force. Magnify the face shown above:



$\vec{n} \equiv$ unit normal pointing from inside to out.

The stress tensor, T^{ij}, is the force per area

$$T^{ij} = \text{Force in the } j\text{-th direction per area in the } i\text{-th}$$

or

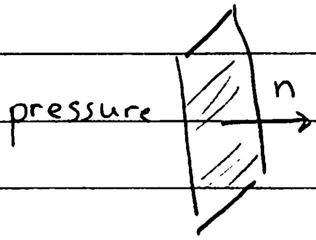
$$n_i T^{ij} = \text{Force in the } j\text{-th direction by the inside on the outside.}$$

$-n_i T^{ij}$ is the force of outside on inside.

In general, there are lots of forces, mechanical (pressure) in addition to electrical.

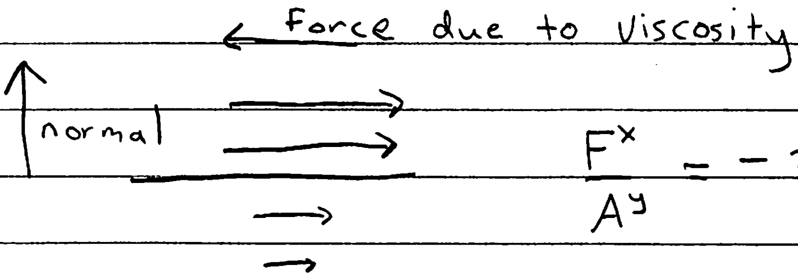
Examples

Ideal Fluid: $T^{ij} = \overset{\text{pressure}}{p}\,\delta^{ij}$



$$\frac{F^x}{A^x} = n_i T^{ix} = T^{xx} = p$$

Viscous Fluid;



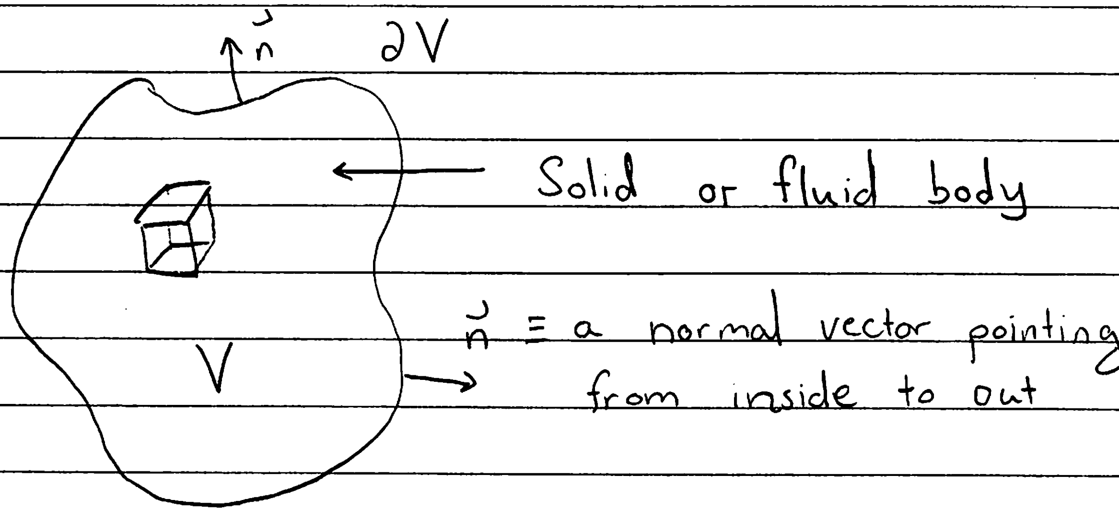
$$\frac{F^x}{A^y} = -\eta \frac{\Delta v^x}{\Delta y} = T^{yx}$$

In electricity we will show that

$$T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

N.B. I use a sign convention opposite from jackson. For jackson, T^{ij} is the force of the outside on the inside. For me, T^{ij} is the force of inside on outside.

Stress Tensor and Momentum Conservation Laws



Take an element of the solid or fluid and ask how the total momentum per volume $\equiv \vec{g}_{tot}$ changes.

We would expect $\vec{g}_{TOT}$ to obey a conservation law:

(1)
$$\frac{\partial g^i_{TOT}}{\partial t} + \partial_i T^{ij} = 0$$

In this case, the total momentum will be conserved:

(2)
$$\frac{\partial P^j_{TOT}}{\partial t} = \frac{\partial}{\partial t}\int_V d^3r\, g^j_{TOT} = -\int d^3r\, \partial_i T^{ij}$$

divergence theorem

$$= -\underbrace{\int_{\partial V} da\, n_i T^{ij}}$$

For a mechanically isolated system this surface integral will vanish.

So to summarize. From Eq (1) we see that

$\partial_t g^j_{TOT}$ = change in momentum/volume/time, also known as force per volume.

and conclude

(1) $$\boxed{-\partial_i T^{ij} = \text{force volume} = f^j}$$

Similarly the net force from Eq. (2) is

(2) $$\boxed{\frac{dP^j_{TOT}}{dt} = -\int da\, n_i T^{ij}}$$

Forces and Stress in Electrostatics

Consider a charged fluid; the force per volume is

$$f^j = \rho E^j$$

This should be the divergence of something, $f^j = -\partial_i T^{ij}$

$$f^j = (\partial_i E^i) E^j \qquad \nabla \cdot E = \rho$$

in class problem see next page for solution

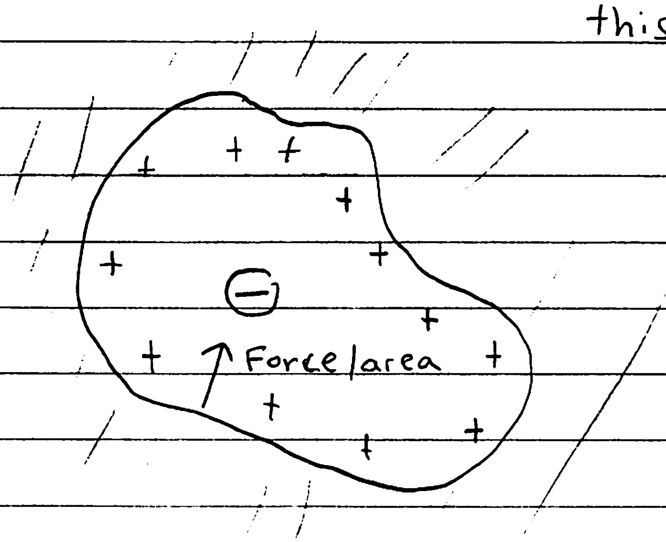
$$= \partial_i \underbrace{\left(E^i E^j - \frac{1}{2} \delta^{ij} E^2\right)}_{-T^{ij}}$$

So

$$\boxed{T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}}$$

Previously we derived that the force/area on a conductor wall is $F = \sigma^2/2$. Derive this result using the stress tensor.

E = 0 inside metal



Solution to In Class Problems:

(1) $f^j = \rho E^j$

$$\nabla\cdot E = \rho \qquad \nabla\times E = 0$$

$$\partial_i E^i = \rho \qquad \partial_i E_j - \partial_j E_i = 0$$

f^j should be the divergence of something:

$f^j = -\partial_i T^{ij} \longrightarrow$ what is T^{ij}?

Solution:

$$f^j = \rho E^j$$

$$= (\partial_i E^i) E^j$$

$$= \partial_i (E^i E^j) - E^i \underbrace{\partial_i E^j}$$

from $\partial_i E^j - \partial^j E_i = 0$.
i.e. $\nabla\times E = 0$

$$= \partial_i (E^i E^j) - E^i \partial^j E_i$$

$$= \partial_i (E^i E^j) - \frac{1}{2}\partial^j (E^i E_i)$$

$\frac{1}{2}\partial^j (E^i E_i) = E^i \partial^j E_i$

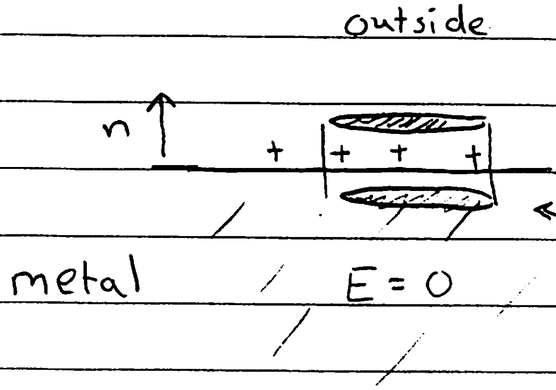
relabel $E^i E_i = E^2 = E^k E_k$

$$= \partial_i (E^i E^j - \frac{1}{2}\delta^{ij} E^2)$$

use $\partial_i \delta^{ij} = \partial^j$

$$= -\partial_i \underbrace{(-E^i E^j + \frac{1}{2}\delta^{ij} E^2)}_{T^{ij}}$$

Force on a metal surface from stress tensor



← gaussian "pillbox"

Now, $\vec{E}_{out} = \sigma \vec{n}$ or $E^j_{out} = \sigma n^j$, i.e.

$$E^j_{out} \equiv E_n n^j .$$

To find the net force/area on a small bit of surface we look at the difference in the force per area between the outside and inside surfaces of the gaussian "pillbox"

$$\text{net force} = -\oint_{\text{pillbox}} da_i \, T^{ij}$$

$$\text{Or} \quad \frac{\text{net force}}{\text{area}} = -n_i \left(T^{ij}_{out} - \cancel{T^{ij}_{in}} \right)$$

$\cancel{T^{ij}_{in}} = 0$ since $E_{in\ metal} = 0$

$$= -n_i \left(-E^i_{out} E^j_{out} + \frac{E^2_{out}}{2} \delta^{ij} \right)$$

use $E^i_{out} = E_n n^i$

$$= -n_i \left(-E_n^2 n^i n^j + \frac{E_n^2}{2} \delta^{ij} \right)$$

use $\vec{n}^2 = 1$ is unit vector

$$= E_n^2 n^j - \frac{E_n^2}{2} n_i \delta^{ij}$$

$$= \frac{E_n^2}{2} n^j$$

$$\frac{\text{net force}}{\text{area}} = \frac{\sigma^2}{2} n^j \Longrightarrow \boxed{\frac{\vec{F}}{A} = \frac{\sigma^2}{2} \vec{n}}$$