Grad, Div, Curl, and Laplacian

CARTESIAN $d\mathbf{l} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ $\quad d^3r = dx\,dy\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$ $\quad d^3r = \rho\,d\rho\,d\phi\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$ $\quad d^3r = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Laplacian

∇^2 acting on components of $\boldsymbol{A}$ (e.g. $\nabla^2 A_z$) indicates the scalar Laplacian in the appropriate coordinate system.

CARTESIAN:

$$\nabla^2 \boldsymbol{A} = \nabla^2 A_x \,\hat{\mathbf{x}} + \nabla^2 A_y \,\hat{\mathbf{y}} + \nabla^2 A_z \,\hat{\mathbf{z}}$$

CYLINDRICAL:

$$\begin{aligned}\nabla^2 \boldsymbol{A} = &\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi}\right)\hat{\boldsymbol{\rho}} \\ &+ \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{\phi}} \\ &+ \nabla^2 A_z \,\hat{\boldsymbol{z}}\end{aligned}$$

SPHERICAL:

$$\begin{aligned}\nabla^2 \boldsymbol{A} = &\left(\nabla^2 A_r - 2\frac{A_r}{r^2} - -\frac{2}{r^2 \sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial A_\phi}{\partial \phi}\right)\hat{\boldsymbol{r}} \\ &+ \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{\phi}} \\ &+ \nabla^2 A_z \,\hat{\boldsymbol{z}}\end{aligned}$$

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

Problem 1. 2D Electrostatics

(a) Determine the potential from a line of charge with uniform charge per length λ in Heavyside-Lorentz units.

(b) Consider a 2D charge distribution $\rho(x, y)$ in a finite region of space satisfying the Poisson equation in two dimensions

$$-\nabla^2\varphi(x, y) = \rho(x, y) \tag{1}$$

Determine the potential at distances far from the charge density, *i.e.* determine the 2D analog of the Cartesian multipole expansion. Develop the expansion to quadrupole order, and define the appropriate Cartesian monopole, dipole, and quadrupole moments.

(c) Consider a 2d "dipole" placed at the center of a (vacuum) cylindrical cavity of radius a, carved out of an infinite block of dielectric material with dielectric constant ϵ (see figure). The dipole is formed by two lines of charge, with charge per length λ and $-\lambda$, separated by an infinitesimal distance d. Determine the potential inside and outside the cavity.

(d) Determine the induced charge per area on the surface of the cavity walls.

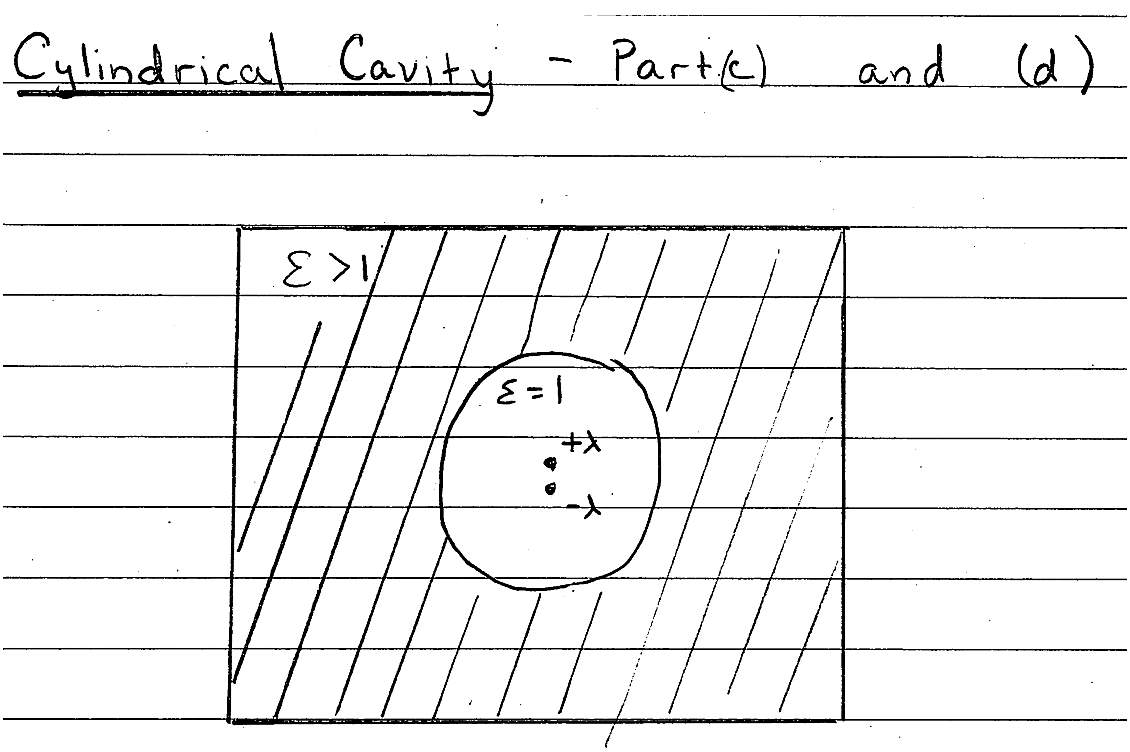

Figure 3: 2D Electrostatics

Problem 2. Decay of a surface current:

A cylindrical shell of radius a has conductivity σ and thickness $\Delta \ll a$. Inside and outside the shell is free space. At time $t = 0$ the shell carries a surface current $\boldsymbol{K}(0, \phi) = \hat{\boldsymbol{z}} K_o \sin\phi$, but at this moment the battery driving this current is switched off. (You may consider the current to be uniform across the shell.)

(a) Determine the coulomb gauge vector potential and magnetic field at time $t = 0$ using a magnetostatic approximation.

(b) What is the electric field in the shell at time $t = 0$. (This requires essentially no computation).

(c) Determine $\boldsymbol{K}(t, \phi)$ at subsequent times using a quasi-static approximation. (Hint: determine the induced electric field in the shell due to a time dependent current of the form $\boldsymbol{K}(t, \phi) = \hat{\boldsymbol{z}} K(t) \sin\phi$.)

(d) Estimate the decay time numerically for a centimeter sized copper shell with $\Delta/a \sim 0.1$.

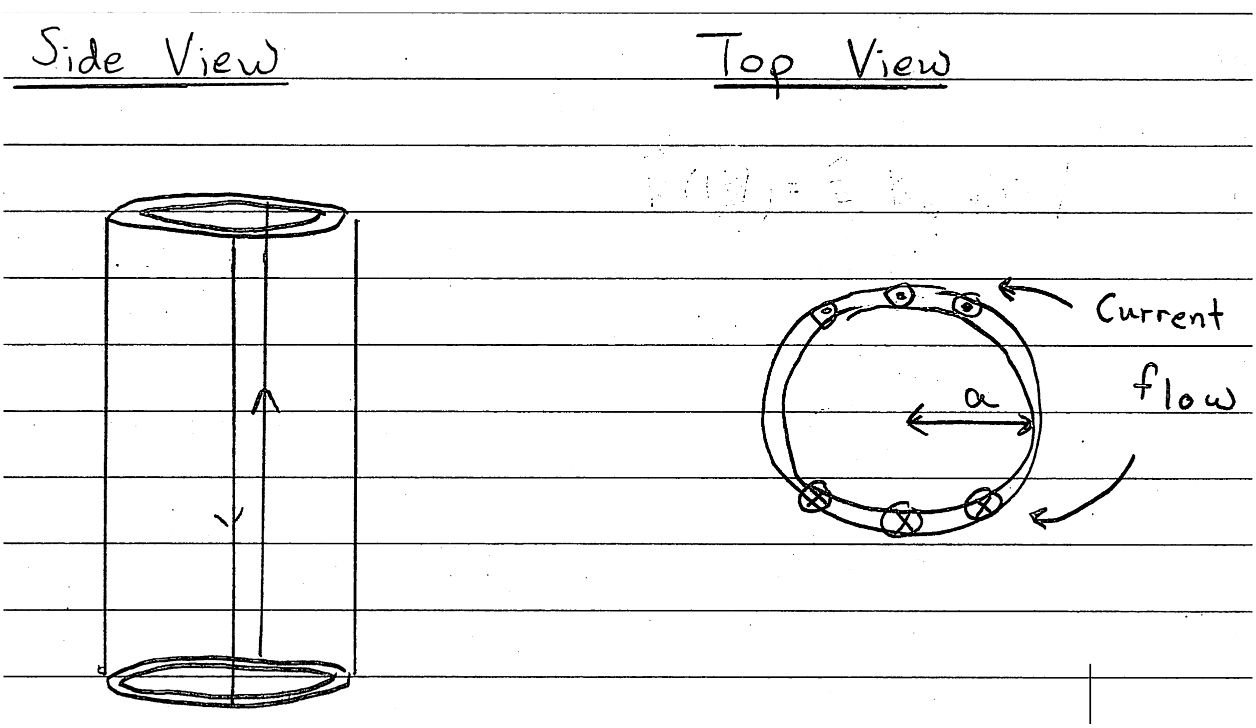

Figure 4: Decay of a current

Problem 3. Snell's law in a crystal

Consider light of frequency ω in vacuum incident upon a uniform dielectric material filling the space $y > 0$. The light is polarized in plane (as shown below) and has incident angle θ_1. The dielectric material has uniform permittivity ϵ and $\mu = 1$.

(a) Derive Snell's law from the boundary conditions of electrodynamics.

Consider light propagating in a crystal with $\mu = 1$ and dielectric tensor ϵ_{ij}. Along the principal crystalline axes ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \tag{2}$$

and thus, along the axes $D_i = \epsilon_i E_i$ (no sum over i).

(b) Starting directly from the Maxwell equations in the dielectric medium, show that the frequency and wave numbers of the plane wave solutions $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ in the crystal are related by

$$\det\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \epsilon_i}{c^2}\delta_{ij}\right) = 0 \qquad \text{(no sum over } i\text{)}. \tag{3}$$

Now consider light of frequency ω in vacuum incident upon a dielectric crystal. The light has incident angle θ_1, and propagates in the $x - y$ plane, *i.e.* $k_z = 0$. The incident light is also polarized in $x - y$ plane, and the axes of the dielectric crystal are aligned with the x, y, z axes (see below). Only the y axis of the crystal has a slightly larger dielectric constant than the remaining two axes,

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon\,(1+\delta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{4}$$

with $\delta \ll 1$.

(c) Determine angle of refraction (or $\sin\theta_2$) including the first order in δ correction to Snell's law.

(d) Is the refracted angle smaller or larger than in the isotropic case? Explain physically. Does the angular dependence of your correction makes physical sense? Explain physically.

(e) If the incident light is polarized along the z axis (out of the $x - y$ plane), what is the deviation from Snell's law? Explain physically.

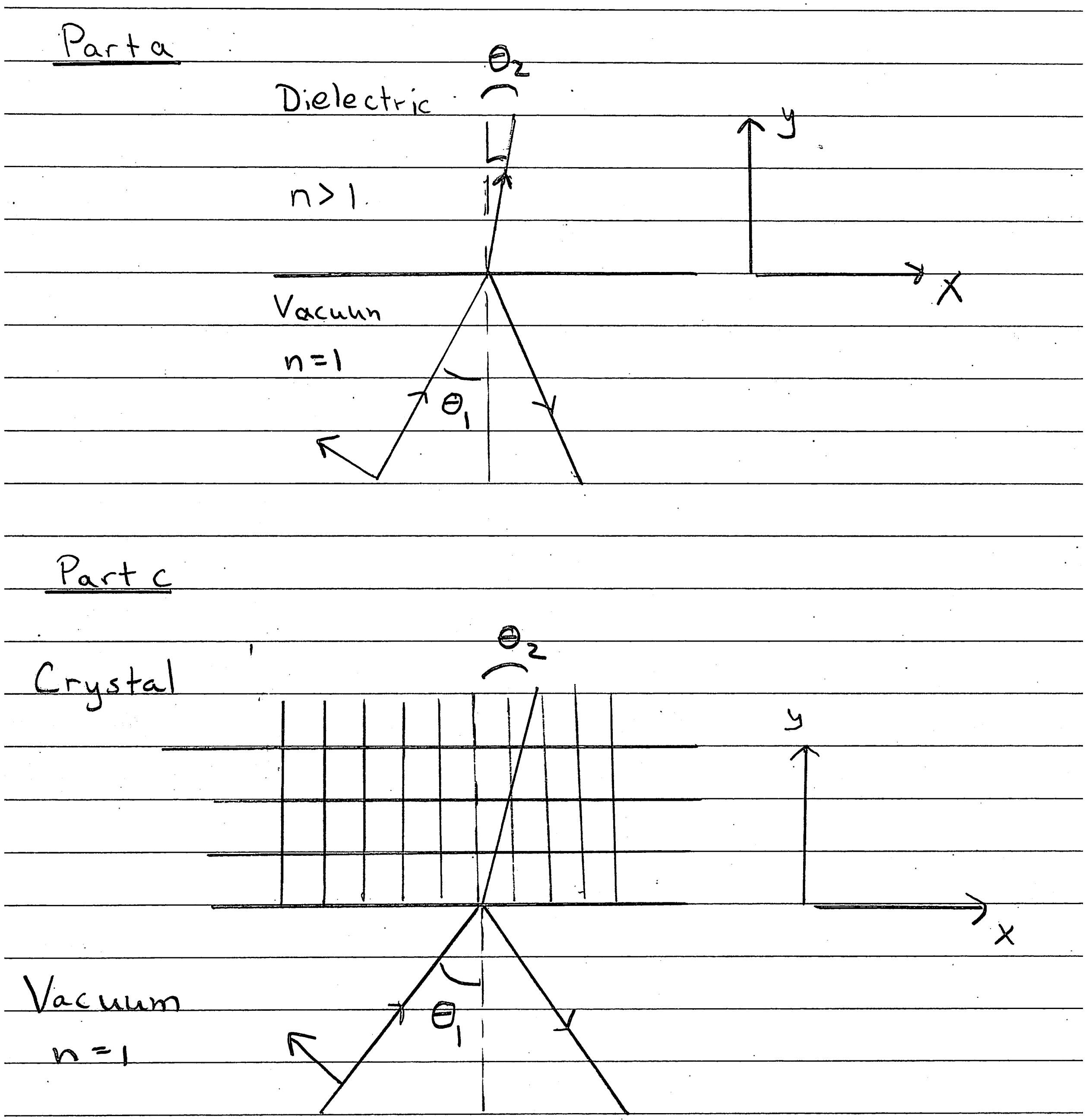

Figure 5: Snell's law geometry