Problem 1. 2D Electrostatics

(a) Determine the potential from a line of charge with uniform charge per length λ in Heavyside-Lorentz units.

(b) Consider a 2D charge distribution $\rho(x, y)$ in a finite region of space satisfying the Poisson equation in two dimensions

$$-\nabla^2\varphi(x, y) = \rho(x, y) \tag{2}$$

Determine the potential at distances far from the charge density, *i.e.* determine the 2D analog of the Cartesian multipole expansion. Develop the expansion to quadrupole order, and define the appropriate Cartesian monopole, dipole, and quadrupole moments.

(c) Consider a 2d "dipole" placed at the center of a (vacuum) cylindrical cavity of radius a, carved out of an infinite block of dielectric material with dielectric constant ϵ (see figure). The dipole is formed by two lines of charge, with charge per length λ and $-\lambda$, separated by an infinitesimal distance d. Determine the potential inside and outside the cavity.

(d) Determine the induced charge per area on the surface of the cavity walls.

Cylindrical Cavity - Part (c) and (d)

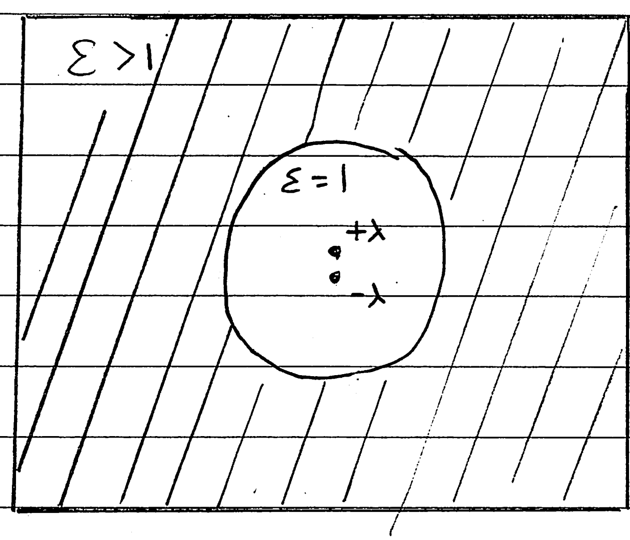
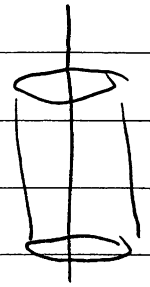


Figure 3: 2D Electrostatics

2D Electrostatics

a)



$$\int \vec{E} \cdot d\vec{a} = Q$$

$$2\pi \rho \, E_\rho \, L = \lambda L$$

$$E_\rho = \frac{\lambda}{2\pi \rho}$$

So

$$\varphi = -\int E_\rho \, d\rho = -\frac{\lambda}{2\pi} \log \rho + C$$

b) Then in 2D

$$\varphi(\vec{r}) = \int d^2\vec{r}_0 \; \rho(r_0) \; \frac{-1}{2\pi} \log |\vec{r} - \vec{r}_0|$$

So we expand for $|\vec{r}| \gg r_0$

$$|\vec{r} - \vec{r}_0| = (r^2 + r_0^2 - 2 r \cdot r_0)^{1/2}$$

$$= r \left(1 + \frac{r_0^2}{r^2} - 2 \hat{r} \cdot \frac{\vec{r}_0}{r} \right)^{1/2}$$

So

$$\log |\vec{r} - \vec{r}_0| = \log r + \frac{1}{2} \log \left(1 + \frac{r_0^2}{r^2} - 2 \hat{r} \cdot \frac{\vec{r}_0}{r} \right)$$

2D Electrostatics

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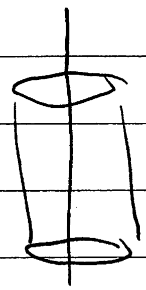

$$\int \vec{E} \cdot d\vec{a} = Q$$

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So

$$\log|\vec{r} - \vec{r}_0| = \log r + \frac{1}{2} \log \left(1 + \frac{r_0^2}{r^2} - 2\hat{r} \cdot \frac{\vec{r}_0}{r}\right)$$

Using

$$\log(1+x) = x - \frac{x^2}{2}$$

Find

$$\log|\vec{r} - \vec{r}_0| = \log r + \frac{1}{2}\left(-2\hat{r}\cdot\frac{\vec{r}_0}{r} + \frac{r_0^2}{r^2} - \frac{1}{2}\frac{(-2\hat{r}\cdot\vec{r}_0)^2}{r^2}\right)$$

So

$$\varphi(r) = \int_{r_0} \rho(\vec{r}_0)\left[-\frac{1}{2\pi}\log r + \frac{1}{2\pi}\frac{\hat{r}\cdot\vec{r}_0}{r} + \frac{1}{2\pi r^2}\left(\frac{(\hat{r}\cdot\vec{r}_0)}{r^2} - \frac{r_0^2}{2r^2}\right)\right]$$

So

$$\varphi(r) = -\frac{\lambda}{2\pi}\log r + \frac{1}{2\pi}\frac{\hat{r}\cdot\vec{p}}{r} + \frac{1}{2\pi}\hat{r}_i\hat{r}_j\frac{Q^{ij}}{r^2}$$

here

$$\lambda = \int d^2r_0\, \rho(r_0)$$

$$\vec{p} = \int d^2r_0 \, \rho(\vec{r}_0) \, \vec{r}_0$$

$$Q^{ij} = \int d^2r_0 \, \rho(r_0) \left(r_0^i r_0^j - \frac{r_0^2}{2} \delta^{ij} \right)$$

Note that $Q^{ij} \delta_{ij} = 0$ since δ^{ij} runs with two dimensions $\delta^{ij}\delta_{ij} = 2$.

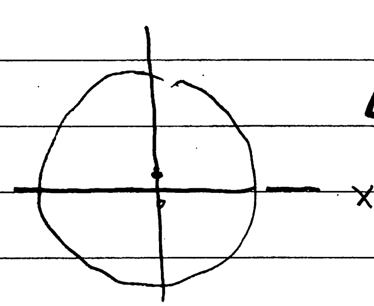
c) Then for this part we expand the potential inside and out. The "dipole" at the center has strength

$$\vec{p} = \lambda\left(\frac{d}{2}\hat{y}\right) + (-\lambda)\left(-\frac{d}{2}\hat{y}\right) = \lambda d \, \hat{y}$$

Thus we expect the potential to be asymptotic with a dipole form:

$$\lim_{\rho \to 0} \varphi = \frac{1}{2\pi \rho} \hat{\rho} \cdot (\lambda d \, \hat{y}) = \frac{\lambda d \sin\phi}{2\pi \rho}$$

see coordinates



Then the general solution to laplace eq. is:

$$\varphi = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty}\left(A_n \rho^n + \frac{B_n}{\rho^n}\right)\sin n\phi$$

$$+ \text{ cos terms}$$

The source has $n=1$ $\propto \sin\phi$, so we limit the solution to the $n=1$ components

$$\varphi_{in} = \left(A\rho + \frac{B}{\rho}\right)\sin\phi$$

Then the requirement that

$$\varphi \xrightarrow[\rho\to 0]{} \frac{\lambda d \sin\phi}{2\pi\rho}$$

gives

$$\boxed{B = \frac{\lambda d}{2\pi}}$$

To find A we need to find the solution outside and match b.c.

$$\varphi_{out} = \left(\cancelto{0}{\tilde{A}\rho} + \frac{\tilde{B}}{\rho}\right)\sin\phi$$

$$\varphi_{out} = \frac{\tilde{B}}{\rho}\sin\phi$$

Continuity gives

$$\varphi_{out}\Big|_a = \varphi_{in}\Big|_a$$

Note: this is the same as requiring $E_{\parallel}^{out} - E_{\parallel}^{in} = 0$

$$(\star) \quad Aa + \frac{B}{a} = \frac{\tilde{B}}{a}$$

Then we have the B.C.

$$\vec{n}\cdot(\vec{D}_{out} - \vec{D}_{in}) = 0$$

So using $\vec{n}\cdot\vec{D} = \varepsilon(-\partial\varphi/\partial\rho)$ we have:

$$-\varepsilon \frac{\partial \varphi_{out}}{\partial\rho}\Big|_{\rho=a} - \left(-\frac{\partial\varphi_{in}}{\partial\rho}\right)\Big|_{\rho=a} = 0$$

Or

$$\frac{\varepsilon\tilde{B}}{a^2} + A - \frac{B}{a^2} = 0 \qquad (\star\star)$$

Solving Eqs $(\star)$ and $(\star\star)$ for A and $\tilde{B}$

So

$$A = -\frac{(\varepsilon-1)}{\varepsilon+1}\frac{B}{a^2} \qquad \tilde{B} = \frac{2B}{1+\varepsilon}$$

So

$$\varphi_{in} = \left[-\left(\frac{\varepsilon-1}{\varepsilon+1}\right) \frac{\lambda d \rho}{2\pi a^2} + \frac{\lambda d}{2\pi \rho} \right] \sin\phi$$

$$\varphi_{out} = \frac{2}{1+\varepsilon} \frac{\lambda d}{2\pi\rho} \sin\phi$$

d) The charge

0 (vacuum)

$$\sigma = -\vec{n} \cdot (\vec{P}_2 - \cancel{\vec{P}_1})$$

$$\sigma_p = -\vec{n} \cdot \vec{E}_2 \, (\varepsilon - 1)$$

Then $\vec{E} = -\nabla\varphi$

$$\sigma_p = (\varepsilon - 1) \left. \frac{\partial \varphi_{out}}{\partial \rho} \right|_{\rho = a}$$

So

$$\sigma_p = - \frac{2(\varepsilon-1)}{(\varepsilon+1)} \left. \frac{\lambda d}{2\pi\rho^2} \sin\phi \right|_a$$

$$= - \frac{2(\varepsilon-1)}{(\varepsilon+1)} \frac{\lambda d}{2\pi a^2} \sin\phi$$

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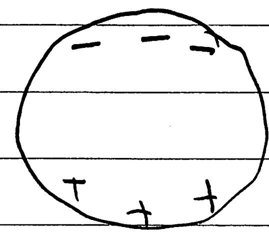
As we expect the charge is negative on the top half.



Problem 2. Decay of a surface current:

A cylindrical shell of radius a has conductivity σ and thickness $\Delta \ll a$. Inside and outside the shell is free space. At time $t = 0$ the shell carries a surface current $\boldsymbol{K}(0, \phi) = \hat{\boldsymbol{z}} K_o \sin\phi$, but at this moment the battery driving this current is switched off. (You may consider the current to be uniform across the shell.)

(a) Determine the coulomb gauge vector potential and magnetic field at time $t = 0$ using a magnetostatic approximation.

(b) What is the electric field in the shell at time $t = 0$. (This requires essentially no computation).

(c) Determine $\boldsymbol{K}(t, \phi)$ at subsequent times using a quasi-static approximation. (Hint: determine the induced electric field in the shell due to a time dependent current of the form $\boldsymbol{K}(t, \phi) = \hat{\boldsymbol{z}} K(t) \sin\phi$.)

(d) Estimate the decay time numerically for a centimeter sized shell with $\Delta/a \sim 0.1$

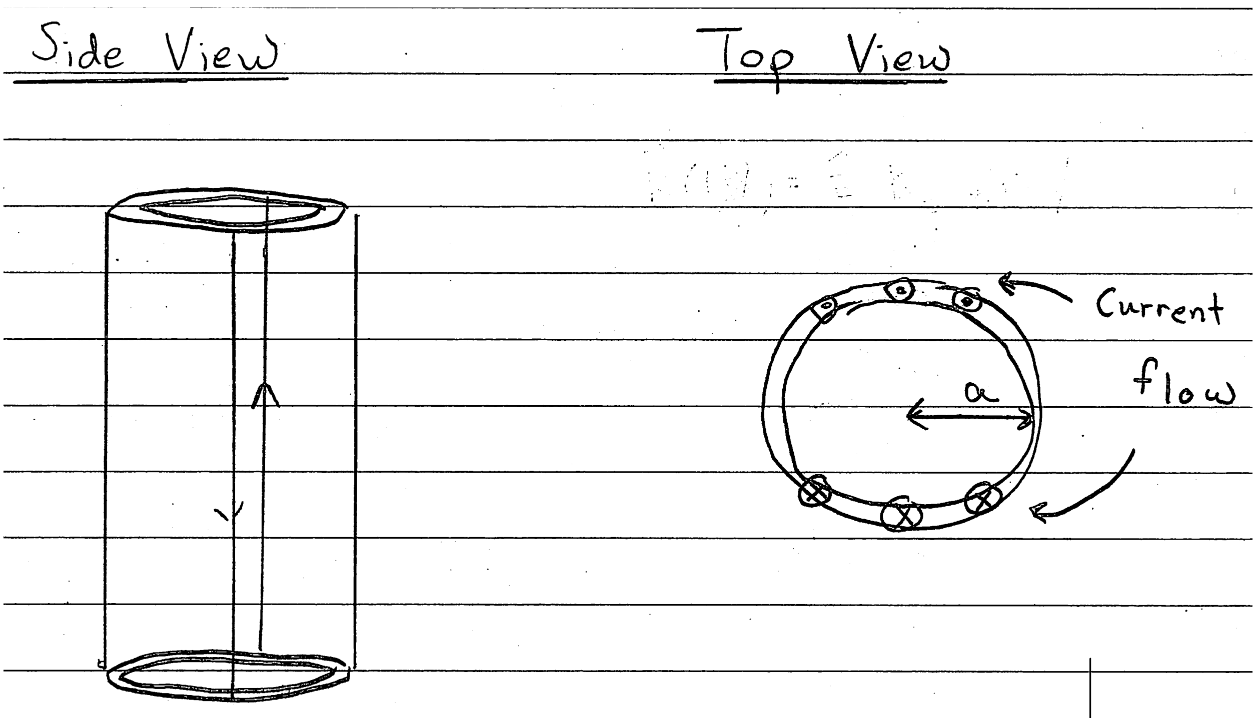


Figure 4: Decay of a current

Decay of Current

a) The potential satisfies

$$-\nabla^2 A^z = 0$$

So the solution inside

$$A^z = D_0 + C_0 \ln\rho + \sum_{n=1}^{\infty} \left(D_n \rho^n + \frac{C_n}{\rho^n} \right) \sin n\phi$$

$$+ \cos n\phi \text{ terms}$$

Then

$$A^z = \left(D_1 \rho + \frac{C_1}{\rho} \right) \sin\phi$$

has the correct azimuthal dependence

$$A^z_{in} = D_1 \rho \sin\phi$$

$$A^z_{out} = \frac{C_1}{\rho} \sin\phi$$

Continuity gives

$$A^z_{in} = C \frac{\rho}{a} \sin\phi$$

$$A^z_{out} = C \frac{a}{\rho} \sin\phi$$

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The magnetic field

$$B_\phi = -\frac{\partial A^z}{\partial \rho} = \begin{cases} -\frac{C}{a}\sin\phi & \rho < a, \\ +\frac{Ca}{\rho^2}\sin\phi & \rho > a \end{cases}$$

$$B_\rho = \frac{1}{\rho}\frac{\partial A^z}{\partial \phi} = \begin{cases} \frac{C}{a}\cos\phi & \rho < a, \\ \frac{Ca}{\rho^2}\cos\phi & \rho > a \end{cases}$$

Then

$$n \times (B_{out} - B_{in}) = \vec{K}/c$$

$$n \cdot (B_{out} - B_{in}) = 0$$

So comparison

$$\frac{2C}{a}\sin\phi = \frac{K_z}{c}\sin\phi \implies \boxed{C = \frac{aK_0/c}{2}}$$

So

$$\boxed{A^z = \begin{cases} \frac{K_0}{2c}\rho\sin\phi \\ \frac{K_0 a^2}{2c\rho}\sin\phi \end{cases}}$$

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b) The current at $t=0$ is

$$\vec{j} = \sigma E$$

$$\frac{\vec{K}}{\Delta} = \sigma E$$

$$\frac{\vec{K}(t=0)}{\Delta\sigma} = \vec{E}(t=0)$$

c) Taking the induced field

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \cancel{\nabla\phi}$$

with $\vec{E} = \frac{\vec{K}}{\Delta\sigma}$ and $A^z(t, \rho=a) = \frac{K(t)}{2c} a \sin\phi$

one has

$$K^z(t,\phi) = -\frac{\Delta\sigma a}{c} \partial_t K(t) \frac{\sin\phi}{2c}$$

So with $\vec{K}(t,\phi) = \hat{z}\, K(t) \sin\phi$

$$-K(t) = \frac{a\Delta\sigma}{2c^2} \partial_t K(t)$$

So

$$K(t) = K_0 e^{-\frac{2c^2}{\sigma}\frac{t}{\Delta a}}$$

d) Recognizing $D = \frac{c^2}{\sigma} \equiv$ magnetic diffusion coefficient

$D \sim \frac{cm^2}{\text{millisec}}$ for Cu

For $a \sim 10\,cm$ $\Delta \sim 1\,cm$

$\tau_{life} \sim \frac{\Delta a}{2D} \sim$ millisecs

Problem 3. Snell's law in a crystal

Consider light of frequency ω in vacuum incident upon a uniform dielectric material filling the space $y > 0$. The light is polarized in plane (as shown below) and has incident angle θ_1. The dielectric material has uniform permittivity ϵ and $\mu = 1$.

(a) Derive Snell's law from the boundary conditions of electrodynamics.

Consider light propagating in a crystal with $\mu = 1$ and dielectric tensor ϵ_{ij}. Along the principal crystalline axes ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \tag{3}$$

and thus, along the axes $D_i = \epsilon_i E_i$ (no sum over i).

(b) Starting directly from the Maxwell equations in the dielectric medium, show that the frequency and wave numbers of the plane wave solutions $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ in the crystal are related by

$$\det\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \epsilon_i}{c^2}\delta_{ij}\right) = 0 \qquad \text{(no sum over } i\text{)}. \tag{4}$$

Now consider light of frequency ω in vacuum incident upon a dielectric crystal. The light has incident angle θ_1, and propagates in the $x - y$ plane, *i.e.* $k_z = 0$. The incident light is also polarized in $x - y$ plane, and the axes of the dielectric crystal are aligned with the x, y, z axes (see below). Only the y axis of the crystal has a slightly larger dielectric constant than the remaining two axes,

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon\,(1+\delta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{5}$$

with $\delta \ll 1$.

(c) Determine angle of refraction (or $\sin\theta_2$) including the first order in δ correction to Snell's law.

(d) Is the refracted angle smaller or larger than in the isotropic case? Explain. Does the angular dependence of your correction makes physical sense? Explain.

(e) If the incident light is polarized along the z axis (out of the $x - y$ plane), what is the deviation from Snell's law? Explain.

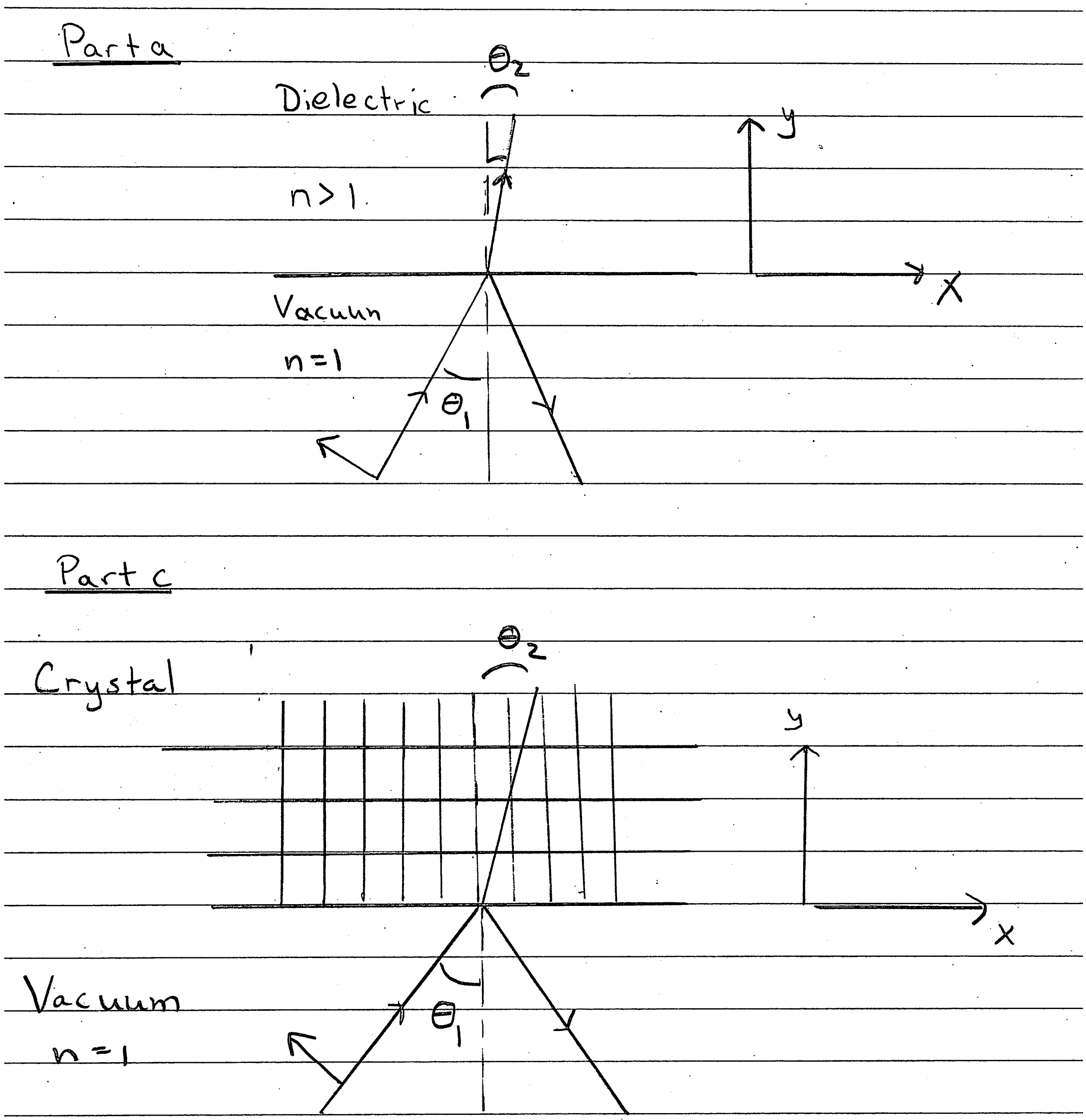
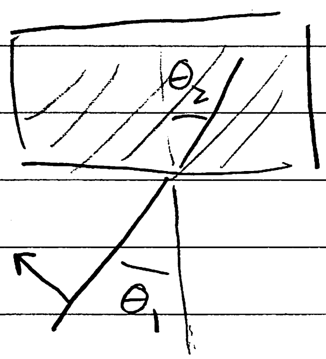


Figure 5: Snell's law geometry

Snell's Law pg. 1

a)



The incident plane wave with $|\vec{k}_R| = |\vec{k}_I| = \omega/c$ has:

$$\vec{E} = \vec{E}_I e^{i\vec{k}_I\cdot\vec{r} - i\omega t} + \vec{E}_R e^{i\vec{k}_R\cdot\vec{r} - i\omega t}$$

Then the transmitted wave:

$$\vec{E} = E_T e^{i\vec{k}_T\cdot\vec{r} - i\omega t} \qquad |\vec{k}_T| = n_2\frac{\omega}{c}$$

The equality of phases at interface, is needed for b.c:

$$i\vec{k}_I\cdot\vec{r} - i\omega t\Big|_{y=0} = +i\vec{k}_T\cdot\vec{r} - i\omega t\Big|_{y=0}$$

gives:

$$ik_I^x\Big| = ik_T^x\Big| \qquad -i\omega_I = -i\omega_T$$

So

$$\sin\theta_1 = \frac{k_x}{k_I} \qquad \sin\theta_2 = \frac{k_T^x}{k_T}$$

$$\sin\theta_1 = \frac{ck_x}{\omega} \qquad \sin\theta_2 = \frac{c}{n\omega} \implies \boxed{\frac{1}{n_2}\sin\theta_1 = \sin\theta_2}$$

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b) $\nabla \cdot D = \rho$

$\nabla \times B = J/c + \frac{1}{c}\partial_t D$

$\nabla \cdot B = 0$

$-\nabla \times E = \frac{1}{c}\partial_t B$

In free space $\rho = J/c = 0$ then for plane wave e.g.

$\vec{E} = \vec{E}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$, find:

$ik \cdot D = 0$

$i\vec{k} \times \vec{B} = -\frac{i\omega}{c} D$

$i\vec{k} \cdot \vec{B} = 0$

$-ik \times \vec{E} = -\frac{i\omega}{c} B$

So

$ik \times (-i\vec{k} \times \vec{E}) = -\frac{i\omega}{c} i\vec{k} \times B$

$\vec{k}(\vec{k}\cdot\vec{E}) - k^2 \vec{E} = -\frac{i\omega}{c}\left(-\frac{i\omega}{c}\vec{D}\right)$

Snell's Law pg. 3

So we have with $D_i = \varepsilon_i E_i$ (no i sum)
that

$$k_i (k_j E_j) - k^2 E_i = -\frac{\omega^2}{V_i^2} E_i \qquad V_i^2 \equiv \frac{c^2}{\varepsilon_i}$$

Or

$$\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2}{V_i^2} \delta_{ij}\right) E_j = 0 \qquad (\text{Eq} \bigstar)$$

This will only have non-trivial solutions if the determinant is non-zero, by the theory of linear equations:

$$\det \left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \delta_{ij}}{V_i^2}\right) = 0$$

c) The angle of refraction is derived following the methods part a). From the equality of phases

$$k^x = k^x_I = k^x_T \qquad \omega_I = \omega_R = \omega$$

Now we can write out Eq ☆:

$$\begin{pmatrix} -k_y^2 + \frac{\omega^2}{V_x^2} & k_x k_y & 0 \\ k_x k_y & -k_x^2 + \frac{\omega^2}{V_x^2} & 0 \\ 0 & 0 & -k^2 + \frac{\omega^2}{V_x^2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = ($$

Taking the determinant:

$$D = \left(-k^2 + \frac{\omega^2}{V_x^2}\right)\left(\left(-k_y^2 + \frac{\omega^2}{V_x^2}\right)\left(-k_x^2 + \frac{\omega^2}{V_y^2}\right) - k_x^2 k_y^2\right)$$

$$D = \left(-k^2 + \frac{\omega^2}{V_x^2}\right)\left(-\frac{\omega^2 k_x^2}{V_x^2} - \frac{\omega^2 k_y^2}{V_y^2} + \frac{\omega^4}{V_x^2 V_y^2}\right)$$

Setting $D = 0$ we have two solutions:

$$\left(-k^2 + \frac{\omega^2}{V_x^2}\right) = 0$$

← we will return to this in part e). The eigenvector $\vec{E}_T$ is polarized in the z direction. See matrix

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and

$$\left(-\frac{\omega^2}{V_x^2} - \frac{\omega^2}{V_y^2} k_y^2 + \frac{\omega^4}{V_x^2 V_y^2} \right) = 0$$

In this case we must set $E_z = 0$ (see matrix) and the eigenvector is polárized in plane

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writing $V_x^2 = \frac{c^2}{\varepsilon}$, $V_y^2 = \frac{c^2}{\varepsilon(1+\delta)}$ and solving

for k_y from the second term in braces:

$$(k^y)^2 = \frac{\omega^2}{V_x^2} - \frac{V_y^2}{V_x^2} k_x^2$$

now we know K_x and k_y so we know the angle $\tan\theta_R = \frac{k^y}{k^x}$

The rest is algebra

Writing

$$(k^y)^2 = \frac{\omega^2}{V_x^2} - \frac{\varepsilon}{\varepsilon(1+\delta)} k_x^2$$

$$(k^y)^2 = \left(\frac{\omega^2}{V_x^2} - k_x^2\right) + \delta k_x^2$$

So defining $k_{(0)} \equiv \omega/(c/n)$ (i.e. the normal thing)

$$k_x^2 + k_y^2 = k^2 = k_{(0)}^2 + \delta k_x^2 \Rightarrow k = k_{(0)} + \frac{1}{2}\delta \frac{k_x^2}{k_{(0)}}$$

So

$$\sin\theta_R = \frac{k^x}{k} = \frac{k^x}{\left(k_{(0)} + \frac{1}{2}\delta\frac{k_x^2}{k_{(0)}}\right)}$$

$$\simeq \frac{k^x}{k_{(0)}}\left(1 - \frac{\delta}{2}\frac{k_x^2}{k_{(0)}^2}\right)$$

$$\sin\theta_R = \sin\theta_{(0)}\left(1 - \frac{1}{2}\delta \sin^2\theta_{(0)}\right)$$

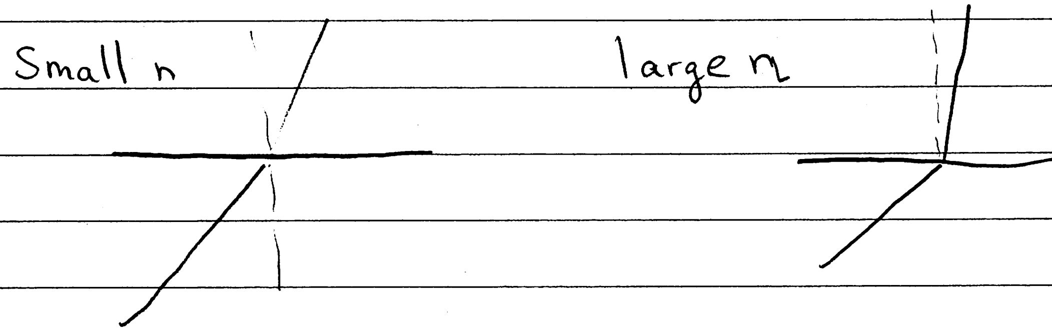
where $\sin\phi_{(0)}$ is the "normal/usual" refraction angle

Using the result from part (a)

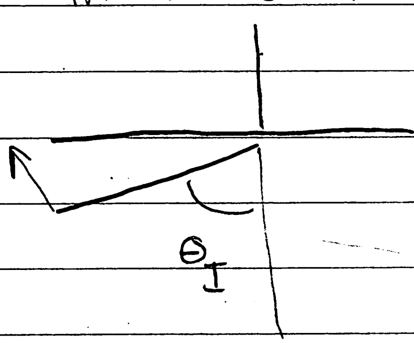
$$\boxed{\sin\theta_R = \frac{1}{n}\sin\theta_I\left(1 - \frac{\delta}{2}\,\frac{\sin^2\theta_I}{n^2}\right)}$$

d) The refracted angle is smaller, this make sense the index of refraction is larger in one direction

For a larger n in the isotropic case the refraction angle is smaller



The larger is the angle of incidence the more the transmitted light is polarized along the slow axis (larger index of refraction axis), i.e.



For larger θ_I the electric field is more polarized along y, so the index of refraction is larger than normal

This explains the larger shift at larger incidence, θ_I

e) If the electric field is in the z-direction, then it is aligned along a single axis of the crystal. we have simply:

$$k^2 = \frac{\omega^2}{V_z^2}$$

This ray then obeys the ordinary snell's law with the index being the index in the z-direction.

$$\sin\theta_2 = \frac{k^x}{k} = \frac{k^x}{\sqrt{\varepsilon_z}\,(\omega/c)} =$$

$$\sin\theta_2 = \frac{1}{n}\sin\theta_1 \qquad n \equiv \frac{1}{\sqrt{\varepsilon}}$$

Reflection from glass 8

So

$$T_2^{zz} = \frac{1}{2}\delta^{zz}\varepsilon E^2 + \frac{1}{2}\delta^{zz} B^2$$

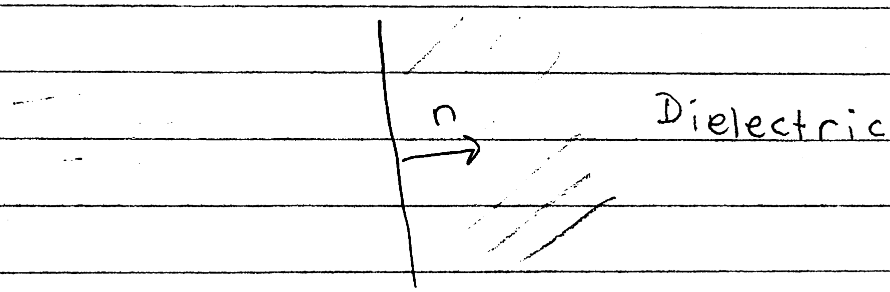
$$T_2^{zz} = \frac{1}{4} E_I^2 \varepsilon t^2 + \frac{1}{4} E_I^2 \varepsilon t^2$$

$$\boxed{T_2^{zz} = \frac{1}{2} E_I^2 n T}$$

$\sqrt{\varepsilon}\, t^2 \equiv T$

$\sqrt{\varepsilon} \equiv n$

So the The difference in the stress tensor tells us about the force



$$-n_a \cdot (T_2^{ab} - T_1^{ab}) = \text{Force in the } b\text{-th direction}$$

$$F^z_{net} = +T_1^{zz} - T_2^{zz}$$

$$F^z = \frac{1}{2} E_I^2 \left[+(1+R) - nT \right] = -\frac{1}{2} E_I^2 \left(\frac{n-1}{n+1} \right)$$

= Away from dielectric!