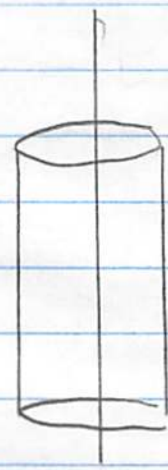
Problem 1 - Wedge pg. 1

a) $\varphi(\rho)$ comes from Gaus Law



$$\int E \cdot d\vec{a} = Q$$

$$E_\rho \, 2\pi \rho L = \lambda L$$

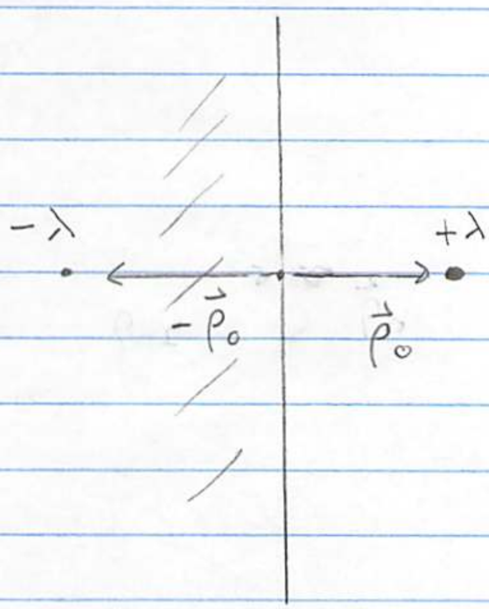
$$E_\rho = \frac{\lambda}{2\pi\rho}$$

$$\varphi = -\int^{\rho} E_\rho \, d\rho$$

$$\boxed{\varphi = \frac{-\lambda}{2\pi} \log \rho + C}$$

b)

Use images:



$$\varphi = -\frac{\lambda}{2\pi} \log |\vec{\rho} - \vec{\rho}_0|$$

$$+ \frac{\lambda}{2\pi} \log |\vec{\rho} + \vec{\rho}_0|$$

here $\vec{\rho}_0 = \rho_0 \hat{x}$

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Expanding the log

$$|\vec{\rho} - \vec{\rho}_0| = (\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\phi)^{1/2}$$

$$= \rho \left(1 + \frac{\rho_0^2}{\rho^2} - 2\frac{\rho_0}{\rho}\cos\phi\right)^{1/2}$$

$$\simeq \rho \left(1 - \frac{\rho_0}{\rho}\cos\phi\right)$$

So

$$\varphi \simeq -\frac{\lambda}{2\pi}\log\rho - \frac{\lambda}{2\pi}\log\left(1 - \frac{\rho_0}{\rho}\cos\phi\right)$$

$$+ \frac{\lambda}{2\pi}\log\rho + \frac{\lambda}{2\pi}\log\left(1 + \frac{\rho_0}{\rho}\cos\phi\right)$$

$$\boxed{\varphi \simeq \frac{2\lambda\rho_0}{2\pi\rho}\cos\phi}$$

we used $\log(1+x) \simeq x$

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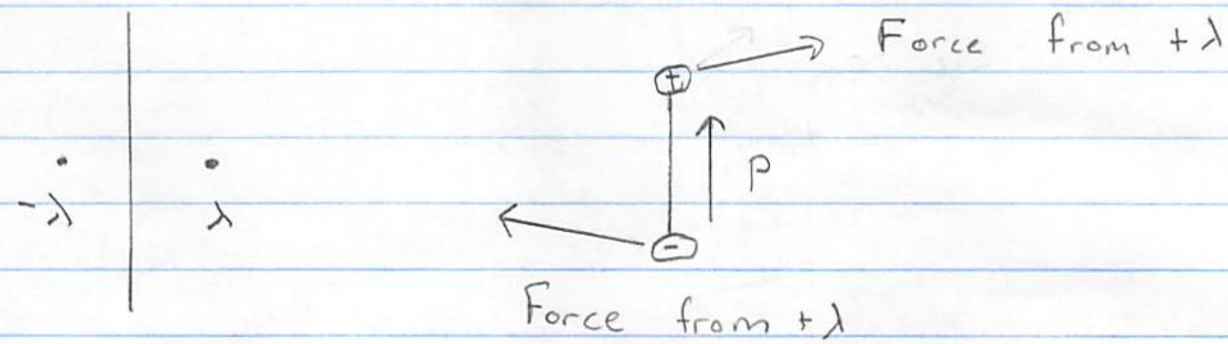
c) Now consider a dipole and draw forces:



i) Drawing the forces from the $+\lambda$ of charge there is a net y-component of force. From the $-\lambda$ of charge there is a negative y-component of force, but it is weaker because the $-\lambda$ charge is farther away. Thus, the force is in the y-direction.

ii) Using

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = p_0 \frac{\partial}{\partial y} \vec{E}$$

Then

$$\varphi = \frac{2\lambda \rho_0}{2\pi \rho} \cos\phi \qquad \rho = \sqrt{x^2 + y^2}$$

$$\vec{E} = -\nabla \varphi = -\frac{\partial \varphi}{\partial \rho}\hat{\rho} + \frac{-1}{\rho}\frac{\partial \varphi}{\partial \phi}\hat{\phi}$$

$$= \frac{2\lambda \rho_0}{2\pi \rho^2}\cos\phi\,\hat{\rho} + \frac{2\lambda}{2\pi\rho^2}\sin\phi\,\hat{\phi}$$

Then using

$$\hat{\rho} = \cos\phi \;\hat{x} + \sin\phi \;\hat{y}$$

$$\hat{\phi} = -\sin\phi \;\hat{x} + \cos\phi \;\hat{y}$$

We find

$$\vec{E} = \frac{2\lambda \rho_0}{2\pi \rho^2} \cos 2\phi \;\hat{x} + \frac{2\lambda \rho_0}{2\pi \rho^3} \sin 2\phi \;\hat{y}$$

$$\vec{E} \simeq \frac{2\lambda \rho_0}{2\pi x^2} \;\hat{x} + \frac{2\lambda \rho_0}{2\pi x^2} \;\frac{2y}{x} \;\hat{y}$$

So

$$\boxed{\vec{F} = p_0 \frac{\partial \vec{E}}{\partial y} = p_0 \left(\frac{2\lambda \rho_0}{2\pi}\right) \frac{2}{x^3} \;\hat{y} \qquad x \equiv \rho}$$

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d) Now separate variables

$$-\nabla^2 \varphi = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\varphi + \frac{-1}{\rho^2}\frac{\partial^2 \varphi}{\partial \phi^2} = 0$$

Consider $-\rho^2 \frac{\nabla^2 \varphi}{\varphi} = 0$

This gives with $\varphi_m = R(\rho)\,\Phi(\rho)$

$$-\frac{\rho}{R}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R + \underbrace{\frac{-1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}_{\text{this is a constant call it } m^2} = 0$$

Now let

$$-\frac{\partial^2 \Phi_m}{\partial \phi^2} = m^2 \Phi_m \quad \text{and} \quad \left(\frac{-1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2}\right)R = 0$$

Solving $\Phi_m = A_m \cos m\phi + B_m \sin m\phi$, $R_m = \left(C\rho^m + \frac{D}{\rho^m}\right)$

The b.c. of the wedge are

$$\Phi_m(-\beta) = \Phi_m(\beta) = 0.$$

This can happen if $m = \ell \frac{\pi}{2\beta}$ $\quad \ell = 1, 2, 3, \ldots$

For ℓ odd we have:

$$\Phi_\ell = \begin{cases} \cos \frac{\ell\pi}{2\beta} & \ell = 1, 3, 5 \ldots \\ \sin \frac{\ell\pi}{2\beta} & \ell = 2, 4, 6 \ldots \end{cases}$$

ii) Then we need to normalize our functions

$$\int_{-\beta}^{\beta} d\phi \; \Phi_\ell(\phi)\, \Phi_{\ell'}(\phi) = (\text{Const})\, \delta_{\ell\ell'}$$

Then to determine the constant

$$\text{Const} = \int_{-\beta}^{\beta} d\phi \quad \cos\left(\frac{\ell\pi}{2\beta}\phi\right) \cos\left(\frac{\ell\pi}{2\beta}\phi\right) \qquad \text{define } x \equiv \frac{\ell\pi}{2\beta}\phi$$

$$= \frac{2\beta}{\ell\pi} \int_{-\frac{\ell\pi}{2\beta}\beta}^{(\ell\pi/2\beta)\beta} dx \quad (\cos^2 x) \qquad \langle \cos^2 \rangle = \frac{1}{2}$$

$$= \frac{2\beta}{\cancel{\ell\pi}} \quad 2 \cdot \left(\frac{\cancel{\ell\pi}}{2}\right) \cdot \frac{1}{2}$$

$$= \beta$$

So we define

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Normalized functions $\hat{\Phi}(\phi) \equiv \Phi(\phi)/\sqrt{\beta}$

$$\int_{-\beta}^{\beta} d\phi \; \hat{\Phi}_{\ell}(\phi)\,\hat{\Phi}_{\ell'}(\phi) = \delta_{\ell\ell'}$$

And completeness

$$\boxed{\sum_{\ell} \hat{\Phi}_{\ell}(\phi)\,\hat{\Phi}_{\ell}(\phi') = \delta(\phi-\phi')}$$

iii) Now we solve the potential is

$$-\nabla^2 \varphi = \frac{\lambda}{\rho}\,\delta(\rho-\rho_0)\,\delta(\phi)$$

We have writing

this is $\frac{1}{\sqrt{\beta}}$ for ℓ odd, 0 otherwise

$$\varphi = \sum_{\ell} g_{\ell}(\rho,\rho_0)\,\hat{\Phi}_{\ell}(\phi)\,\hat{\Phi}_{\ell}(0)$$

We find

$g_{\ell}(\rho)$ satisfies

$$\left(\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} g_{\ell}(\rho) + \frac{m^2}{\rho^2}\right) g_{\ell}(\rho,\rho_0) = +\frac{\lambda}{\rho}\,\delta(\rho-\rho_0)$$

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Inside $\rho < \rho_0$

$$g_\ell = A\left(\frac{\rho}{\rho_0}\right)^m$$

Outside $\rho > \rho_0$

$$g_\ell = A\left(\frac{\rho_0}{\rho}\right)^m$$

The jump gives at ρ_0 with out/in $= \rho_0 \pm \varepsilon$

$$-\rho\frac{\partial g_\ell}{\partial\rho}\bigg|_{out} + \rho\frac{\partial g_\ell}{\partial\rho}\bigg|_{in} = \lambda$$

$$2mA = \lambda$$

$$A = \frac{\lambda}{2m} \quad \text{now} \quad m = \frac{\ell\pi}{2\beta}$$

$$A = \frac{\lambda}{\cancel{2}\left(\frac{\ell\pi}{\cancel{2}\beta}\right)} = \frac{\lambda\beta}{\ell\pi}$$

Now then

$$\Psi(\rho,\phi) = \sum_{\ell\ \text{odd}} \frac{\lambda\beta}{\ell\pi}\ \overset{\hat{\Phi}_\ell(\phi)}{\frac{\cos\left(\frac{\ell\pi\phi}{2\beta}\right)}{\sqrt{\beta}}}\ \overset{\hat{\Phi}_\ell(0)}{\frac{1}{\sqrt{\beta}}}\ \left(\frac{\rho_<}{\rho_>}\right)^{\frac{\ell\pi}{2\beta}}$$

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So

$$\boxed{\varphi = \sum_{l=1,3,5\ldots} \left(\frac{\rho_<}{\rho_>}\right)^{l\pi/2\beta} \frac{\lambda}{l\pi} \cos\left(\frac{l\pi}{2\beta}\phi\right)}$$

iv) Now then as $\rho \to \infty$ $\rho_> = \rho$ $\rho_< = \rho_0$
we keep $l = 1$

$$\boxed{\varphi \simeq \left(\frac{\rho_0}{\rho}\right)^{\pi/2\beta} \frac{\lambda}{\pi} \cos\left(\frac{\pi}{2\beta}\phi\right)}$$

Now for $\beta = \pi/2$ we find

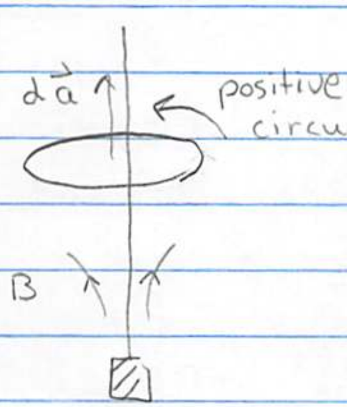
$$\varphi = \frac{\lambda \rho_0}{\pi} \frac{\cos\phi}{\rho}$$ in agreement with part (b)

Problem 2 A Dipole and a ring pg. 1

a) The flux at $t = -\infty$ is zero. If the flux changes then this would induce a net voltage around the loop driving a infinite current. As such the total magnetic flux through the loop must remain zero at all times.

The total magnetic flux has two sources. The flux due to the dipole, and the self flux $\Phi = LI$ which must cancel the flux due to the dipole.

b) The flux from the dipole is computed as follows. First we set some conventions. Positive circulation is counter-clockwise, from above



$$\Phi_B^{dipole} = \int \vec{B} \cdot d\vec{a}$$

$$= \oint \vec{A} \cdot d\vec{l}$$

$$\Phi_B^{dipole} = A_\phi \, 2\pi a$$

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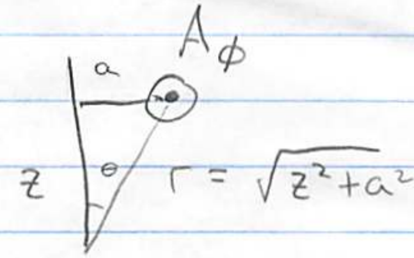
Now

$$\vec{A} = \frac{\vec{m} \times \hat{r}}{4\pi r^2}$$

$$A_\phi = \frac{m \sin\theta}{4\pi r^2}$$



So from geometry:

$$\sin\theta = \frac{a}{r}$$

$$A_\phi = \frac{m a}{4\pi r^3}$$

$$A_\phi = \frac{m a}{4\pi (z^2+a^2)^{3/2}}$$

$$\Phi_B^{\text{dipole}}(t) = \frac{m a}{4\pi ((v_0 t)^2 + a^2)^{3/2}}$$

Now the total flux is zero

c) $$\Phi_{\text{self}} + \Phi_{\text{dipole}} = 0$$

$$L I + \frac{m a}{4\pi ((v_0 t)^2 + a^2)^{3/2}} = 0$$

So

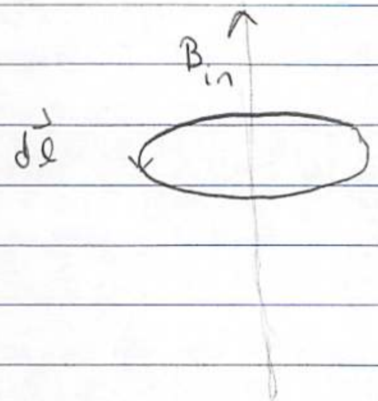
$$\boxed{I = \frac{-m a / L}{4\pi ((v_0 t)^2 + a^2)^{3/2}}}$$

Clockwise current from above

$I < 0$

Problem 3 - The magnetic field in a conducting tube

a) From the def, $\vec{B} = \nabla \times \vec{A}$, we have



$$\int \vec{B} \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$$

So

$$B_{in}(t)\, \pi \rho^2 = A_\phi \, 2\pi\rho$$

$$\boxed{\frac{1}{2} B_{in}(t)\, \rho = A_\phi}$$

This clearly satisfies the coulomb gauge condition $\nabla \cdot A = 0$ since $\partial_\phi A^\phi = 0$

ii) The electric field is

$$\vec{E} = -\frac{1}{c} \partial_t \vec{A}$$

So

$$E_\phi(t) = -\frac{1}{2} \frac{\rho}{c} \dot{B}_{in}(t)$$

$$\boxed{E_\phi(t) = + \frac{i\omega\rho}{2c} B_{in} e^{-i\omega t}}$$

Use:

b.i) $$\nabla \times \vec{B} = \frac{\vec{J}}{c}$$

And $\vec{J} = \sigma \vec{E}$

$$\nabla \times B = \frac{\sigma}{c} \vec{E}$$

$$\nabla \times \nabla \times B = \frac{\sigma}{c} \nabla \times E$$

$$\vec{\nabla}(\nabla \cdot B) - \nabla^2 \vec{B} = \frac{\sigma}{c^2}\left(-\partial_t B\right)$$

So

$$\frac{c^2}{\sigma} \nabla^2 \vec{B} = \partial_t \vec{B}$$

and the diffusion coefficient is c^2/σ

b.ii) Now we try to solve the diffusion equation. Substituting $\vec{B}(x,t) = e^{-i\omega t + ikx}\,\hat{z}\,B$

$$\frac{c^2}{\sigma}(-k^2) = -i\omega$$

$$k = \pm\sqrt{i}\sqrt{\frac{\sigma\omega}{c^2}} = \pm\frac{(1+i)}{\delta} \quad \text{with } \delta \equiv \sqrt{\frac{2c^2}{\sigma\omega}}$$

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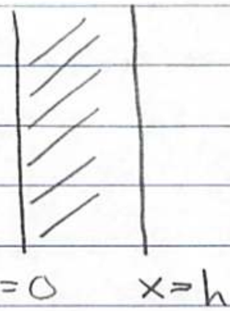
b. iii) inside the metal take the magnetic field in the z-direction

$$B_z(t,x) = B_z(x)\, e^{-i\omega t}$$

Then is a gen. solution

decreasing increasing

$$B_z(x) = C e^{ikx} + D e^{-ikx}$$



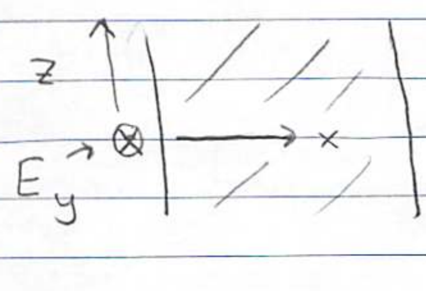
here $k = \frac{(1+i)}{\sqrt{2}}$

We have boundary conditions

$$B_z(x=0) = \boxed{B_{in} = C + D}$$

$$B_z(x=h) = \boxed{B_0 = C e^{ikh} + D e^{-ikh}}$$

We have three unknowns C, D, B_{in} and only two equations. The remaining equation come from the electric field on the inner interface



This electric field is related to B_{in}.

$$(\nabla \times B)_y = \left(\frac{J_y}{c}\right)$$

$$-\frac{\partial B_z(x)}{\partial x} = \frac{\sigma}{c} E_y$$

Now at the interface E_y is related to B_{in} by the results of part (a), and there we showed at $\rho = a$

$$E_y \Big|_{\substack{x=0 \\ \text{or } \rho = a}} = \frac{i\omega a}{2c} B_{in}$$

So we find

$$-ik\,(C - D) = \frac{\sigma}{c}\left(\frac{i\omega a}{2c}\right) B_{in}$$

$$\boxed{k\,(C - D) = -\frac{a}{\delta^2} B_{in}}$$

These three equations are sufficient to determine C, D, and B_{in}

b. iv) Now lets solve the equations for $\delta \ll h \ll a$

Note

$$(k\delta) = 1 + i$$

So

(Eq★) $$(1+i)(C-D) = -\frac{a}{\delta} B_{in} = -\frac{a}{\delta}(C+D)$$

Since $kh \gg 1$ smaller by an exponent

$$B_o = \underbrace{Ce^{ikh}}_{\propto e^{-h/\delta}} + \underbrace{De^{-ikh}}_{e^{+h/\delta}}$$

So to good approximation:

(Eq★★) $$B_o = De^{-ikh} \qquad D = B_o e^{ikh}$$

Solving writing $1+i = \sqrt{2}e^{i\phi_o}$ $\qquad \phi_o = \pi/4$

$$\sqrt{2}e^{i\phi_o}(C-D) = -\frac{a}{\delta}(C+D)$$

$$\left(\sqrt{2}e^{i\phi_o} + \frac{a}{\delta}\right)C = \left(\sqrt{2}e^{i\phi_o} - \frac{a}{\delta}\right)D \qquad a/\delta \gg 1$$

So $C \simeq -D$ up to small corrections, $O\left(\frac{\delta}{a}\right)$