

## Grad, Div, Curl, and Laplacian

**CARTESIAN**     $d\ell = x\hat{x} + y\hat{y} + z\hat{z}$      $d^3r = dx dy dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$


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**CYLINDRICAL**     $d\ell = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$      $d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$


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**SPHERICAL**     $d\ell = dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi}$      $d^3r = r^2 \sin\theta dr d\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\phi}{\partial\theta}\right]\hat{r} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here  $\psi$  is a scalar function and  $\mathbf{A}$  is a vector field.

### Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

### Integral Identities

$$\int_V d^3r \nabla \cdot \mathbf{A} = \int_S dS \hat{\mathbf{n}} \cdot \mathbf{A}$$

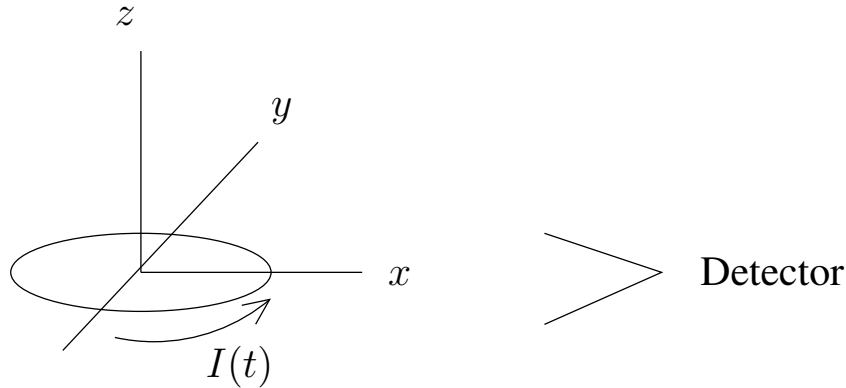
$$\int_V d^3r \nabla \psi = \int_S dS \hat{\mathbf{n}} \psi$$

$$\int_V d^3r \nabla \times \mathbf{A} = \int_S dS \hat{\mathbf{n}} \times \mathbf{A}$$

$$\int_S dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_C d\ell \cdot \mathbf{A}$$

$$\int_S dS \hat{\mathbf{n}} \times \nabla \psi = \oint_C d\ell \psi$$

Figure 2: Vector and integral identities. Here  $\psi$  is a scalar function and  $\mathbf{A}$ ,  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are vector fields.



### Problem 1. Radiation from a circular wire

An antenna consists of a circular loop of current of radius  $a$  located in the  $x$ - $y$  plane with its center at the origin,

$$I(t) = I_o \cos(\omega t) = \text{Re } I_o e^{-i\omega t}. \quad (1)$$

We will determine the radiation fields from this antenna.

- Under what conditions can the radiation field be calculated using the multipole expansion? What is the lowest multipole that contributes to the radiation?
- Using the lowest multipole moment approximation, determine the time average power per solid angle  $\overline{dP/d\Omega}$  as measured along the  $x$ -axis.
- Still working in the limit of the lowest multipole, determine the polarization of the radiated field when the radiation is viewed along the  $x$ -axis. Explain your answer using formulas.
- Now ...do not make a multipole expansion. Determine the average power radiated along the  $x$ -axis. (See the integrals below.)
- By expanding the integrand of part (d) as appropriate for the multipole expansion, show that you recover the the result of part (b) for the power per solid angle.

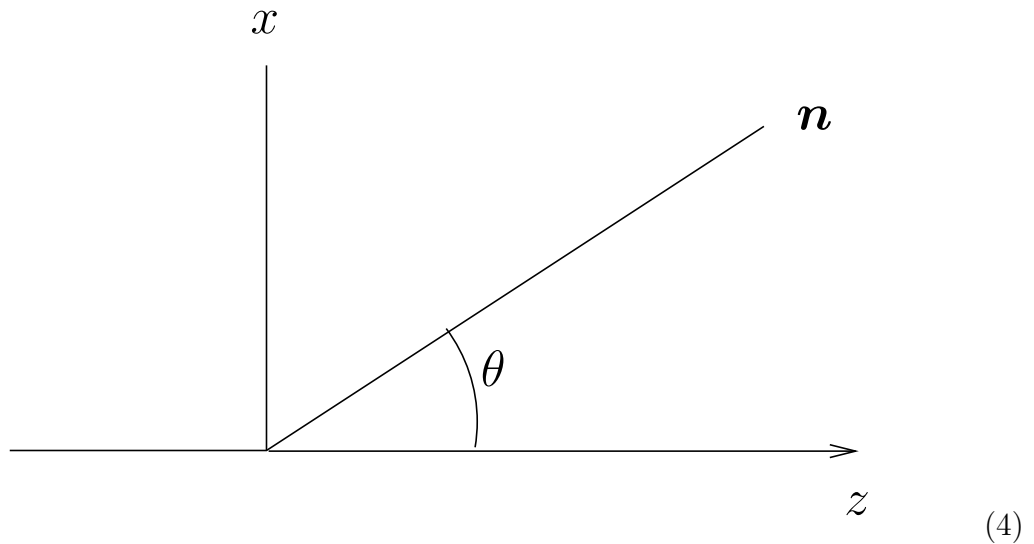
The following integrals are useful

$$\int_0^{2\pi} du \cos(nu) e^{-ix \cos(u)} = 2\pi (-i)^n J_n(x) \quad (2)$$

$$\int_0^{2\pi} du \sin(nu) e^{-ix \cos(u)} = 0 \quad (3)$$

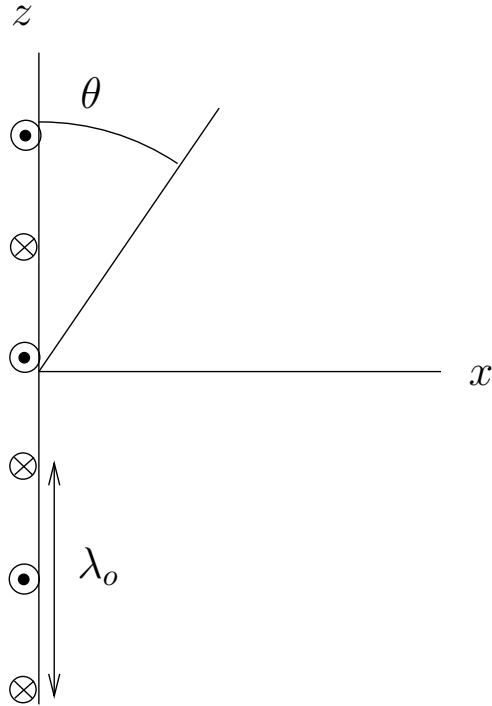
## Problem 2. Scattering from an electron

- (a) Write down all Maxwell equations for the electric and magnetic fields in covariant form. In particular, covariantly show that the source free Maxwell equations are automatically satisfied, provided the field strength  $F^{\mu\nu}$  is related to  $A^\mu$  in the appropriate way. Show how the equations for the gauge potential  $A^\mu = (\varphi, \mathbf{A})$  in the Lorentz gauge can be derived from the remaining (covariant) Maxwell equations.
- (b) Use the equations derived in part (a) (perhaps written non-covariantly) to derive the Larmor-like formula for the radiation potential  $\mathbf{A}_{\text{rad}}$  in the far field from a non-relativistic accelerating charged particle.
- (c) In Thomson scattering, long wavelength unpolarized light is scattered off an electron. Determine the total cross section for this process using Larmor-like results. Express your result in terms of the fine structure constant,  $\alpha \simeq 1/(137)$ , and the electron Compton wavelength.
- (d) Now consider incoming light linearly polarized in  $x$ -direction scattering off an electron at the origin into an angle  $\theta$  as shown below.



Derive the cross section for the polarized light to yield light polarized in the  $z$ - $x$  plane at angle  $\theta$ .

- (e) What is the cross section of part (d) at a scattering angle of  $90^\circ$ ? Give a physical explanation for the cross section at this scattering angle.



### Problem 3. Radiation during lateral acceleration

A charged relativistic point particle of mass  $m$  moves with average velocity  $v$  along the  $z$  axis. The particle is weakly accelerated in the  $y$ -direction (in and out of the page) by a spatially dependent electric field of wavelength  $\lambda_o$  (see above)

$$E^y(z) = E_o \cos(k_o z), \quad k_o = \frac{2\pi}{\lambda_o}. \quad (5)$$

The force is small, *i.e.* the particle moves essentially in a straight line at constant  $v$ , but the  $y$  component of the acceleration is non-zero.

- (a) Determine the acceleration as a function of time to leading order in  $E_o$ . Assume that the pitch the particle's trajectory (*i.e.* the angle between the velocity and the  $z$ -axis) is negligibly small.
- (b) Determine the time averaged power emitted per unit solid angle at an angle  $\theta$  in the  $z-x$  plane (see above), *i.e.* determine

$$\left. \frac{dW}{dT d\Omega} \right|_{\theta}. \quad (6)$$

- (i) Use

$$1 - \mathbf{n} \cdot \boldsymbol{\beta}(T) \approx \frac{1}{2\gamma^2} + \frac{\theta^2}{2}, \quad (7)$$

to express the angular distribution in the ultra-relativistic limit.

- (ii) Sketch a polar plot of Eq. (6) in the non-relativistic and ultra-relativistic limits.
- (c) What is the (total) time averaged power radiated in the ultra-relativistic limit. (A derivation of the necessary formulas is not required.)
- (d) Using the ultra-relativistic approximation described above, determine the Fourier spectrum of the radiated electric field at an angle  $\theta$  in the  $z-x$  plane, *i.e.* determine

$$\mathbf{E}_{\text{rad}}(\omega, \mathbf{r}). \quad (8)$$

You should find that the spectrum is proportional to a delta-function so that only one frequency is observed at a specified angle. What is that frequency?

- (e) Determine the time averaged frequency spectrum per solid angle in the  $z-x$  plane in the ultra-relativistic limit, *i.e.*

$$\frac{dP}{d\omega d\Omega} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \frac{dW}{d\omega d\Omega}. \quad (9)$$

As an intermediate step show that

$$\lim_{T \rightarrow \infty} \left| \int_{-T/2}^{T/2} dt e^{i\omega t} \right|^2 = T \times 2\pi\delta(\omega), \quad (10)$$

using the integral

$$\int_{-\infty}^{\infty} dx \left( \frac{2 \sin(x/2)}{x} \right)^2 = 2\pi. \quad (11)$$

- (f) Using Lorentz transformations, explain the characteristic frequency as a function of angle.