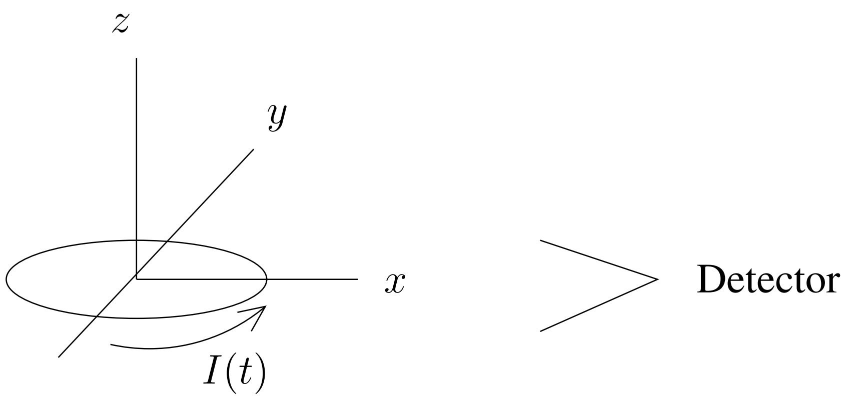


Problem 1. Radiation from a circular wire

An antenna consists of a circular loop of current of radius a located in the $x-y$ plane with its center at the origin,

$$I(t) = I_o \cos(\omega t) = \text{Re}\, I_o e^{-i\omega t} \, . \tag{1}$$

We will determine the radiation fields from this antenna.

(a) Under what conditions can the radiation field be calculated using the multipole expansion? What is the lowest multipole that contributes to the radiation?

(b) Using the lowest multipole moment approximation, determine the time average power per solid angle $\overline{dP/d\Omega}$ as measured along the x-axis.

(c) Still working in the limit of the lowest multipole, determine the polarization of the radiated field when the radiation is viewed along the x-axis. Explain your answer using formulas.

(d) Now …do not make a multipole expansion. Determine the average power radiated along the x-axis. (See the integrals below.)

(e) By expanding the integrand of part (d) as appropriate for the multipole expansion, show that you recover the the result of part (b) for the power per solid angle.

The following integrals are useful

$$\int_0^{2\pi} du\, \cos(nu) e^{-ix\cos(u)} = 2\pi(-i)^n J_n(x) \tag{2}$$

$$\int_0^{2\pi} du\, \sin(nu) e^{-ix\cos(u)} = 0 \tag{3}$$

Problem Circular Wire pg. 1

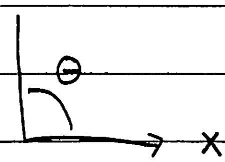
a) $\frac{\omega a}{c} \ll 1$. There is no net e-dipole or e-quad
So the lowest is m-dipole

b) The dipole moment $m = \frac{IA}{c} = \underbrace{\frac{I_0 \pi a^2}{c}}_{\equiv m_0} e^{-i\omega t}$

So for later define $m_0 \equiv \frac{I_0 \pi a^2}{c}$

Then

$$\overline{\frac{dP}{d\Omega}} = \frac{\sin^2\theta}{16\pi^2 c^3} \overline{\ddot{m}^2}$$



$$= \frac{m_0^2 \omega^4}{32\pi^2 c^3} \quad \Bigg) \ \sin\theta = 1$$

c) Using

$$\vec{B} = -n \times \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = n \times n \times \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$\vec{A}_{e\text{-dipole}} = \frac{1}{4\pi r c} \dot{p}(t_e)$$ ← electric dipole case

Circular Wire pg. 2

And thus for an electric-dipole:

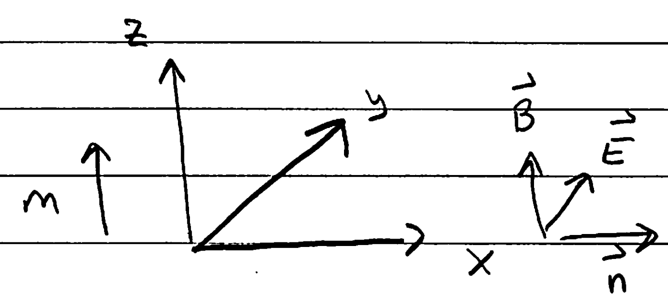
$$\vec{B}_{e\text{-dipole}} = -\frac{\vec{n}}{4\pi r c^2} \times \ddot{\vec{p}}(t_e)$$

So duality allows us to remember:

$$-E_{m\text{-dipole}} = -\frac{\vec{n} \times}{4\pi r c^2} \ddot{m}(t_e)$$

$$E_{m\text{-dipole}} = +\frac{n \times \ddot{m}(t_e)}{4\pi r c^2}$$

Thus if m-points in the z-direction, and n in the x-direction, then E points in the positive y direction, since $\ddot{m}$ has opposite sign of m.



Circular Wire pg. 3

d) Using:

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int d^3 r_0 \; \frac{\vec{J}}{c}(T, r_0)$$

$$= \frac{1}{4\pi r}\int d^3 r \; \frac{\vec{J}}{c}(r_0)\, e^{-i\omega(t - r/c + n\cdot r_0/c)}$$

$$\vec{A}_{rad} = \frac{1}{4\pi r} e^{-i\omega(t - r/c)} \underbrace{\int_{loop} \frac{\vec{I}}{c}\, dl\, e^{-i\omega n\cdot r_0/c}}_{\equiv I_*}$$

Now, $dl = a\, d\phi$, $\vec{I} = -I_0 \sin\phi\, \hat{x} + I_0 \cos\phi\, \hat{y}$,

$\vec{n} = \hat{x}$, $\vec{r}_0 = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$

Then

$$-\frac{i\omega}{c}\vec{n}\cdot\vec{r}_0 = -\frac{i\omega_0 a}{c}\cos\phi$$

So

$$I_* = \int_0^{2\pi} a\, d\phi \left(-\frac{I_0}{c}\sin\phi\,\hat{x} + \frac{I_0}{c}\cos\phi\,\hat{y}\right) e^{-i\omega a/c \cos\phi}$$

$$= -i\frac{I_0 a}{c}\, 2\pi\, J_1(\omega a/c)\;\hat{y}$$

Circular Wire pg. 4

So

$$\vec{A}_{rad} = \frac{1}{4\pi r} e^{-i\omega(t - r/c)} \left[-i I_0 \frac{2\pi a}{c} J_1(\omega a/c) \right] \hat{y}$$

Then

$$\frac{dW}{dT d\Omega} = \frac{c}{2} |r E_{rad}|^2$$

$\uparrow$ time ave

$$E_{rad} = n \times n \times \frac{1}{c} \frac{\partial \vec{A}_{rad}}{\partial t} = -\frac{1}{c} \frac{\partial \vec{A}_{rad-transverse}}{\partial t}$$

So

$$\star \quad \boxed{E_{rad} = \frac{+i\omega}{4\pi r c} e^{-i\omega(t - r/c)} \left(-i I_0 \frac{2\pi a}{c} J_1(\omega a/c) \right) \hat{y}}$$

And

$$\star\star \quad \boxed{\frac{dW}{dT d\Omega} = \frac{c}{32\pi^2} \left(\frac{I_0}{c} \right)^2 \left(\frac{2\pi \omega a}{c} \right)^2 \left(J_1(\omega a/c) \right)^2}$$

Circular Wire pg. 5

Using the integrand of part (d):

$$I_* = \hat{y} \int_0^{2\pi} a\,d\phi \; \frac{I_0}{c} \cos\phi \; e^{-i\omega a/c \cos\phi}$$

And expanding the phase:

$$I_* = \hat{y} \int_0^{2\pi} a\,d\phi \; \frac{I_0}{c} \cos\phi \left(1 - \frac{i\omega a}{c} \cos\phi + \dots\right)$$

$$I_* \simeq \hat{y} \; \frac{-i\omega a^2 2\pi I_0}{c^2} \times \frac{1}{2}$$

$$I_* \simeq \hat{y} \left(-i\omega \frac{m_0}{2}\right)$$

So

$$\vec{A}_{rad} = \frac{e^{-i\omega(t - r/c)}}{4\pi r} \; \frac{-i\omega}{c} \; m_0 \; \hat{y}$$

$$= \frac{1}{4\pi r c^2} \; -\vec{n} \times \dot{\vec{m}}$$ ← this is the dipole formula

We can also return to E_{rad} and replace the term in brackets (which is I) with $\frac{-i\omega m_0}{c}$, leading to (see Eq ★ and ★★):

$$\boxed{\frac{dW}{dT\,d\Omega} = \frac{c}{2} |E_{rad}|^2 = \frac{1}{32\pi^2 c^3} \omega^4 m_0^2}$$

Problem 2. Scattering from an electron

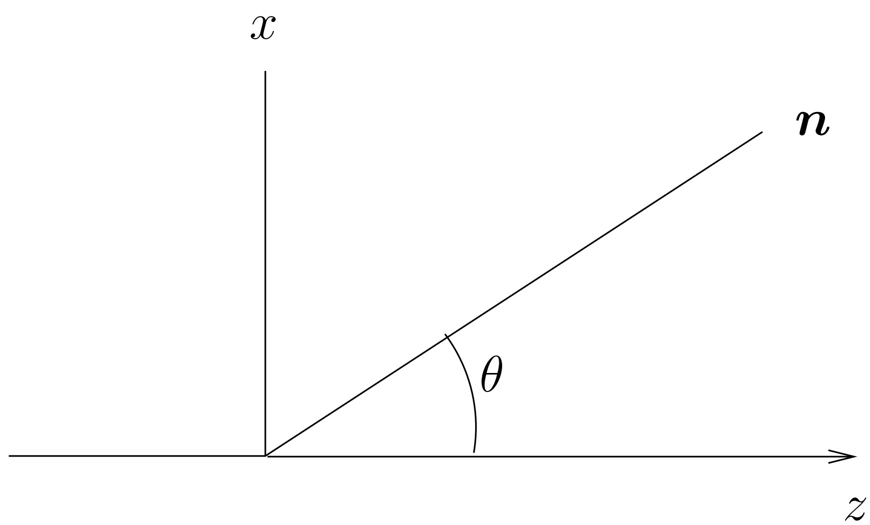
(a) Write down <u>all</u> Maxwell equations for the electric and magnetic fields in covariant form.

In particular, covariantly show that the source free Maxwell equations are automatically satisfied, provided the field strength $F^{\mu\nu}$ is related to A^μ in the appropriate way. Show how the equations for the gauge potential $A^\mu = (\varphi, \boldsymbol{A})$ in the Lorentz gauge can be derived from the remaining (covariant) Maxwell equations.

(b) Use the equations derived in part (a) (perhaps written non-covariantly) to derive the Larmour-like formula for the radiation potential $\boldsymbol{A}_{\rm rad}$ in the far field from a non-relativistic accelerating charged particle.

(c) In Thomson scattering, long wavelength unpolarized light is scattered off an electron. Determine the total cross section for this process using Larmour-like results. Express your result in terms of the fine structure constant, $\alpha \simeq 1/(137)$, and the electron Compton wavelength.

(d) Now consider incoming light linearly polarized in x-direction scattering off an electron at the origin into an angle θ as shown below.



(4)

Derive the cross section for the polarized light to yield light polarized in the $z-x$ plane at angle θ.

(e) What is the cross section of part (d) at a scattering angle of 90^o? Give a physical explanation for the cross section at this scattering angle.

Problem: Scattering pg. 1

a) $-\partial_\mu F^{\mu\nu} = J^\nu / c$

$$\partial_{[\mu} F_{\nu\sigma]} = 0$$ ← totally anti-symmetric combo in μ, ν, σ

Then if $F_{\nu\sigma} = \partial_\nu A_\sigma - \partial_\sigma A_\nu$

$$\partial_{[\mu}\partial_\nu A_{\sigma]} - \partial_{[\mu}\partial_\sigma A_{\nu]} = 0$$

Since $\partial_\mu \partial_\nu A_\sigma$ is symmetric under interchange of μ, ν, but we are anti-symmetrizing with respect to μ, ν we get zero, e.g.

$$\partial_1 \partial_2 A_\sigma - \partial_2 \partial_1 A_\sigma = 0$$

The equations for A follow

$$-\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu / c$$

Lorentz Gauge

$$-\partial_\mu \partial^\mu A^\nu + \partial^\nu (\partial_\mu A^\mu) = J^\nu / c \qquad \partial_\mu A^\mu = 0$$

$$\boxed{-\partial_\mu \partial^\mu A^\nu = J^\nu / c}$$

Scattering pg. 2

b) Using

$$-\Box \vec{A} = \vec{j}/c$$

So

$$\vec{A} = \int \frac{1}{4\pi |\vec{r}-\vec{r}_0|} \frac{\vec{J}(T, r_0)}{c}$$

Where $T = t - \frac{|\vec{r}-\vec{r}_0|}{c}$. Expanding for Large r

$$\vec{A} = \frac{1}{4\pi r} \int d^3 r_0 \frac{\vec{J}}{c}\left(t - \frac{r}{c} + \frac{n \cdot r_0}{c}, r_0\right)$$

$$\simeq \frac{1}{4\pi r} \int d^3 r_0 \frac{\vec{J}}{c}\left(t - \frac{r}{c}, r_0\right)$$

Using $\vec{J} = e\vec{v}(t)\, \delta^3(\vec{r}-\vec{r}_0)$ we find:

$$\vec{A} = \frac{e}{4\pi r} \frac{\vec{v}(t - r/c)}{c}$$

Scattering pg. 3

c) In the Thomson process

$$\vec{a} = \frac{qE_o}{m} e^{-i\omega t}$$

Then the Larmour result

$$P = \frac{2}{3} \frac{q^2}{4\pi} \frac{a^2}{c^3}$$

$$P = \frac{2}{3} \frac{q^2}{4\pi} \frac{q^2}{m^2} \frac{E_o^2}{c^3} \frac{1}{2} \leftarrow \text{time ave}$$

$$\sigma = \frac{P}{c\frac{E_o^2}{2}} = \frac{2}{3} \frac{q^4}{4\pi} \frac{1}{(mc^2)^2}$$

$$= \left(\frac{q^2}{4\pi}\right)^2 \frac{8\pi}{3} \frac{1}{(mc^2)^2}$$

$$\sigma = \alpha^2 \frac{8\pi}{3} \left(\frac{\hbar}{mc}\right)^2$$

d) The incoming light causes an acceleration

$$\vec{a} = \frac{qE_0}{m}\vec{\varepsilon}_0 e^{-i\omega t}$$

Here $\vec{\varepsilon}_0 = \hat{z}$.

Using

$$E_{rad} = \frac{q}{4\pi r c^2} n \times n \times a(t_0)$$

We have:

$$\varepsilon^* \cdot E_{rad} = \frac{q}{4\pi r c^2} \varepsilon^* \cdot (\vec{n}(n \cdot \vec{a}) - \vec{a})$$

$$= \frac{q}{4\pi r c^2} (-\varepsilon^* \cdot a)$$

$$= \frac{q^2}{4\pi r c^2} \left(-\frac{E_0}{m} \varepsilon^* \cdot \varepsilon_0\right)$$

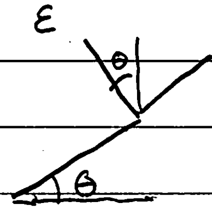
So

$$\frac{dP}{d\Omega} = \frac{c}{2} |r \varepsilon^* \cdot E_{rad}|^2 = \left(\frac{q^2}{4\pi}\right)^2 \frac{c|E_0|^2}{2} \frac{(\varepsilon^* \cdot \varepsilon_0)^2}{(mc^2)^2}$$

So the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\frac{c}{2}|E_0|^2} = \left(\frac{q^2}{4\pi}\right)^2 \frac{(\mathcal{E}^* \cdot \mathcal{E}_0)^2}{(mc^2)^2}$$

So using,

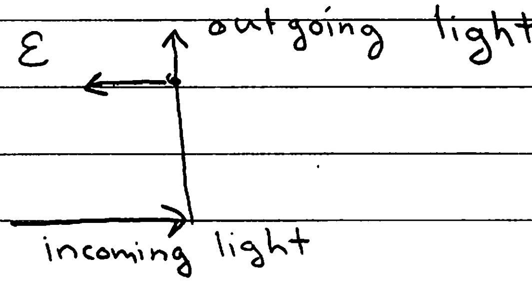


$$\mathcal{E}^* \cdot \mathcal{E}_0 = \cos\theta$$

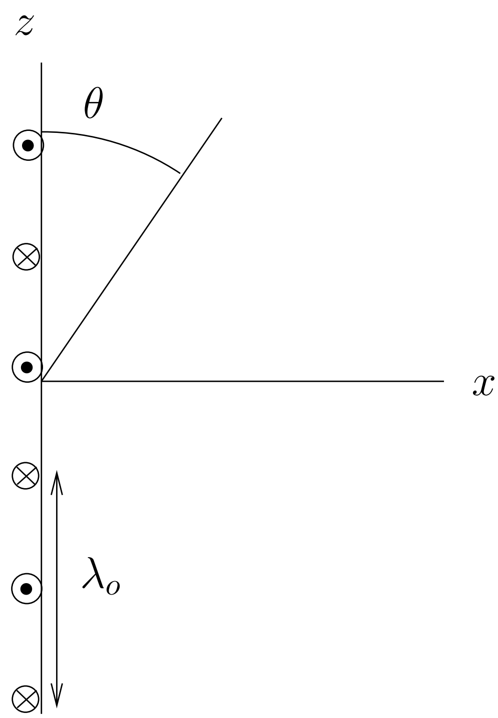
we find

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{4\pi}\right)^2 \frac{\cos^2\theta}{(mc^2)^2}$$

e) The cross section at 90° is zero



This is because in order to get light polarized along $\vec{\mathcal{E}}$, we would need to accelerate the electron in the z-direction (the radiation field is $\propto \mathcal{E} \cdot a$). But the forces caused by the incoming field are in the x-direction. Thus we find no light scattered at 90°.



Problem 3. Radiation during lateral acceleration

A charged relativistic point particle of mass m moves with average velocity v along the z axis. The particle is weakly accelerated in the y-direction (in out of the page) by a spatially dependent electric field of wavelength λ_o (see above)

$$E^y(z) = E_o \cos(k_o z)\,, \qquad k_o = \frac{2\pi}{\lambda_o}\,. \tag{5}$$

The force is small, *i.e.* the particle moves essentially in a straight line at constant v, but the y component of the acceleration is non-zero.

(a) Determine the acceleration as a function of time to leading order in E_o.

(b) Determine the time averaged power emitted per unit solid angle at an angle θ in the $z-x$ plane (see above), *i.e.* determine

$$\left.\frac{dW}{dTd\Omega}\right|_\theta\,. \tag{6}$$

(i) Use

$$1 - \boldsymbol{n}\cdot\boldsymbol{\beta}(T) \equiv \frac{1}{2\gamma^2} + \frac{\theta^2}{2}\,, \tag{7}$$

to express the angular distribution in the ultra-relativistic limit.

(ii) Sketch a polar plot of Eq. (6) in the non-relativistic and ultra-relativistic limits.

(c) What is the (total) time averaged power radiated in the ultra-relativistic limit. (A derivation of the necessary formulas is not required.)

(d) Using the ultra-relativistic approximation described above, determine the Fourier spectrum of the radiated electric field at an angle θ in the $z-x$ plane, *i.e.* determine

$$\boldsymbol{E}_{\mathrm{rad}}(\omega, \boldsymbol{r})\,. \tag{8}$$

You should find that the spectrum is proportional to a delta-function so that only one frequency is observed at a specified angle. What is that frequency?

(e) Determine the time averaged frequency spectrum per solid angle in the $z-x$ plane in the ultra-relativistic limit, *i.e.*

$$\frac{dP}{d\omega d\Omega} \equiv \lim_{T\to\infty} \frac{1}{T}\, \frac{dW}{d\omega d\Omega}\,. \tag{9}$$

As an intermediate step show that

$$\lim_{T\to\infty} \left| \int_{-T/2}^{T/2} dt\, e^{i\omega t} \right|^2 = T \times\, 2\pi\delta(\omega)\,, \tag{10}$$

using the integral

$$\int_{-\infty}^{\infty} dx \left(\frac{2\sin(x/2)}{x} \right)^2 = 2\pi\,. \tag{11}$$

(f) Using Lorentz transformations, explain the characteristic frequency as a function of angle.

Problem Latteral Acceleration

a) We have

$$\frac{d\tilde{p}^y}{dt} = qE^y$$

So

$$\frac{du^y}{dt} = \frac{q}{m} E_0 \cos(k_0 v t)$$

Then

proportional to v^y (small)

$$\frac{du^y}{dt} = \frac{d}{dt}\gamma v^y = \gamma \frac{dv^y}{dt} + v^y \frac{d\gamma}{dt}$$

this is small too. So the whole term is $(\text{small})^2$ and we drop it.

$$\simeq \gamma \frac{dv^y}{dt}$$

So $$a^y \simeq \frac{q E_0}{\gamma m} \cos(k_0 v t)$$

b) The power radiated is

$$\frac{dW}{dT\,d\Omega} = \frac{c}{2} \frac{dt}{dT} |r E_{rad}|^2$$

$$= \frac{c}{2} (1 - n\cdot\beta) |r\cdot E_{rad}|^2$$

time average

Latteral Acceleration

Then

$$E_{rad}^{(t,r)} = \frac{q}{4\pi r c^2} \frac{n \times (n-\beta) \times a(T)}{(1 - n\cdot\beta)^3}$$

So

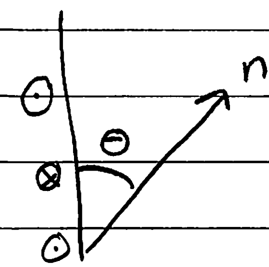
time ave

$$\frac{dW}{dT d\Omega} = \frac{1}{2} \frac{q^2}{16\pi^2 c^3} \frac{|n \times (n-\beta) \times a|^2}{(1 - n\cdot\beta)^5}$$

Now

since $\vec{a} \perp \vec{n}$, $\vec{a}$ out of plane.

$$n \times n \times \vec{a} = -\vec{a} + \vec{n}\,(\cancelto{0}{n\cdot a})$$



and

$$\vec{n} \times (-\vec{\beta}) \times \vec{a} = -\vec{\beta}\,(\cancelto{0}{n\cdot a}) + (n\cdot\beta)\,\vec{a}$$

$$= (n\cdot\vec{\beta})\,\vec{a}$$

So

$$\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a} = -\vec{a}\,(1 - n\cdot\beta)$$

Then

$$\frac{dW}{dT d\Omega} = \frac{q^2}{32\pi^2 c^3} \frac{a^2}{(1 - n\cdot\beta)^3}$$

$$= \frac{q^2}{32\pi^2 c^3} \frac{a^2}{(1 - \beta\cos\theta)^3}$$

(i) Then in the ultra-relativistic limit

$$\frac{1}{(1-\beta\cos\theta)} \sim \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$

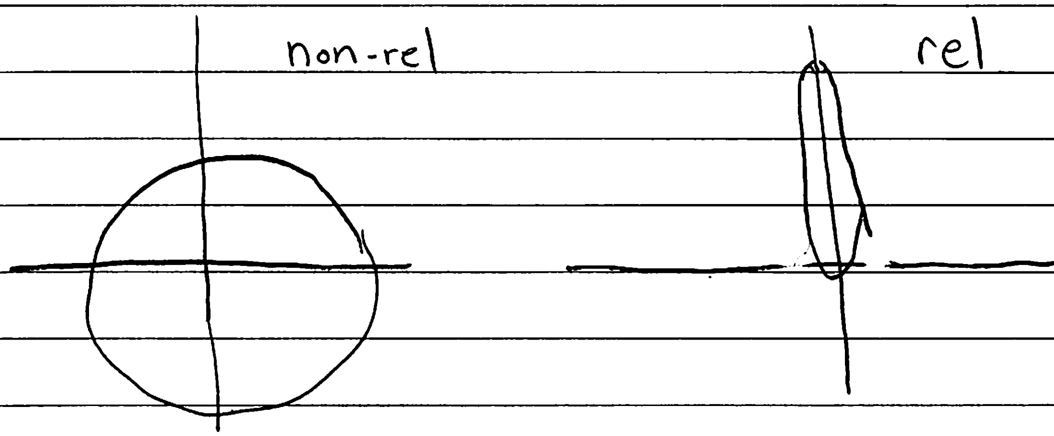
and

$$\frac{dW}{dT\,d\Omega} = \frac{q^2}{4\pi^2 c^3}\,\frac{a^2}{(1+(\gamma\theta)^2)^3}$$

while in the non-relativistic limit

$$\frac{dW}{dT\,d\Omega} = \frac{q^2}{32\pi^2 c^3}\,a^2$$

ii)



Latteral Acceleration pg. 4

c) The total power is given by the Lormour result:

$$P = \frac{e^2}{4\pi}\frac{2}{3c^2}\gamma^4 a_\perp^2 \cdot \frac{1}{2} \leftarrow \text{time averaged accel}$$

$$= \frac{e^2}{4\pi}\frac{2}{3c^3}\gamma^4\left(\frac{qE_0}{\gamma m}\right)^2$$

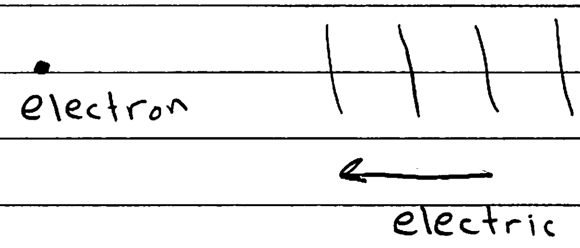
$$= \left(\frac{e^2}{4\pi mc^2}\right)^2 4\pi E_0^2 \gamma^2 c \cdot \frac{2}{3}$$

$$\boxed{P = \sigma_{Thomp}\frac{E_0^2}{2}\gamma^2 c} \qquad \sigma_T = \frac{8\pi}{3} r_e^2 \qquad r_e \equiv \frac{q^2}{4\pi mc^2}$$

Note one could derive this by working in the rest frame of the electron.

electron frame ↙ ↙ lab frame

$$E_0' \simeq \gamma E_0$$



$$\frac{c|E_0'|^2}{2} \simeq \frac{c}{2}\gamma^2 E_0^2$$

d)

Now we compute the radiation spectrum

$$E_{rad}(\omega, r) = \frac{q}{4\pi r c^2} \int_{-\infty}^{\infty} e^{-i\omega(T - \frac{\vec{n}\cdot r_0}{c})} \frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}(T)}{(1 - n\cdot\beta)^2}$$

So using

$$r_0 = \vec{V} T$$

$$\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a} = -\vec{a}(t)(1 - \vec{n}\cdot\vec{\beta}) = -\vec{a}(1 - n\cdot\beta)\ \mathrm{Re}\, e^{-ik_0 V T}$$

Then we have approximating the phase

$$E_{rad}(\omega, r) = \frac{q}{4\pi r c^2} \frac{-\vec{a}}{(1 - n\cdot\beta)} \int_{-\infty}^{\infty} e^{-i\omega T(1 - n\cdot\beta)}\ \mathrm{Re}\, e^{-ik_0 V T}$$

So from $\mathrm{Re}\, z = \frac{z + z^*}{2}$

$$E_{rad}(\omega, r) = \frac{1}{2} \frac{q}{4\pi r c^2} \frac{-\vec{a}}{(1 - n\cdot\beta)} \Big[2\pi\,\delta(\omega(1 - n\cdot\beta) + k_0 V)$$
$$2\pi\,\delta(\omega(1 - n\cdot\beta) - k_0 V) \Big]$$

So the frequencies are at

$$\omega = \frac{\pm k_0 V}{(1 - n\cdot\beta)} = \frac{\pm k_0 V\, 2\gamma^2}{(1 + (\gamma\theta)^2)}$$

e) Latteral Acceleration pg. 6

Then squaring, using the result

$$|2\pi\,\delta(\omega)|^2 = T\;2\pi\,\delta(\omega) \qquad \text{(see below)}$$

We have $2\pi\, dW/d\omega d\Omega = c\,|E_{rad}(\omega)|^2$

$$\frac{2\pi}{T}\frac{dW}{d\omega d\Omega} = \frac{q^2}{64\pi^2c^3}\frac{a^2}{(1-n\cdot\beta)^2}\Big[2\pi\,\delta(\omega(1-n\cdot\beta)+k_0 v) + 2\pi\,\delta(\omega(1-n\cdot\beta)-k_0 v)\Big]$$

Or bringing out $(1-n\cdot\beta)$ from δ-fcn

$$\boxed{\frac{2\pi}{T}\frac{dW}{d\omega d\Omega} = \frac{q^2}{64\pi^2c^3}\frac{a^2}{(1-n\cdot\beta)^3}\Big[2\pi\,\delta(\omega+\omega_*) + 2\pi\,\delta(\omega-\omega_*)\Big]}$$

where $\omega_* = \dfrac{k_0 v}{(1-n\cdot\beta)}$

Note that if we integrate $\displaystyle\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}$ we find:

$$\frac{dW}{dT d\Omega} = \frac{q^2}{32\pi^2c^3}\frac{a^2}{(1-n\cdot\beta)^3}$$ in agreement with part (b)

Proof of Identity

The identity

$$\boxed{\lim_{T\to\infty} \left| \int_{-T/2}^{T/2} e^{-i\omega t}\, dt \right|^2 = T\, 2\pi\, \delta(\omega)}$$

Keep T finite

$$\int_{-T/2}^{T/2} dt\, e^{-i\omega t} = \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{-T/2}^{T/2} = \frac{e^{i\omega T/2} - e^{-i\omega T/2}}{i\omega}$$

$$I = \frac{2 \sin(\omega T/2)}{\omega}$$

Then

$$|I|^2 = \left(\frac{2 \sin \omega T/2}{\omega T} \right)^2 T^2$$

As $T \to \infty$, this is approaching a δ-fcn, $\delta(\omega)$

The weight of the δ-fcn is:

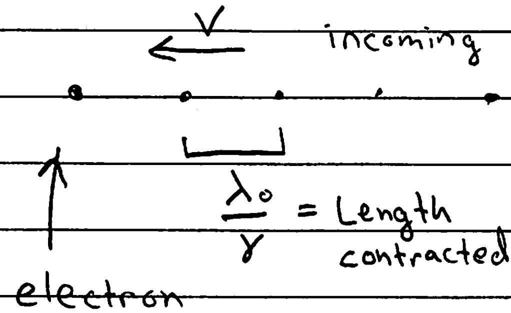
$$\int_{-\infty}^{\infty} d\omega \left(\frac{2 \sin \omega T/2}{\omega T} \right)^2 T^2 = 2\pi T$$

So

$$\lim_{T\to\infty} |I|^2 = T\, 2\pi\, \delta(\omega)$$

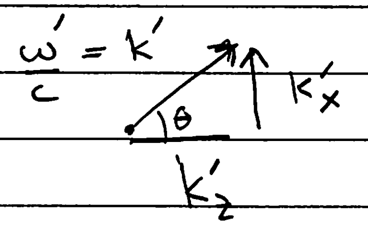
Extra Credit - part (f)

In the frame of the electron, the electron oscillates with a frequency that is given by the length contracted electric field,



$$\text{Period} = \frac{\lambda_0}{\gamma v} \qquad ck' = \omega' = \frac{2\pi}{T} = \frac{2\pi}{\lambda_0} v\gamma = k_0 \gamma v$$

Then due to the forces the electron undergoes dipole radiation $\propto \omega^4$ in all directions, (but not uniformly!



So to find the frequency and wavenumber, we simply need to boost back:

$$k'^{\mu} = \left(\frac{\omega'}{c}, k'_x, 0, k'_z\right)$$

Boosting to the lab frame:

$$\begin{pmatrix} \omega/c \\ k^z \\ k^x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & \\ \gamma\beta & \gamma & \\ & & 1 \end{pmatrix} \begin{pmatrix} \omega'/c \\ k^{z\prime} \\ k^{x\prime} \end{pmatrix}$$

$$\omega/c = \gamma\,\omega'/c + \gamma\beta\, k^{z\prime}$$

$$k^z = \gamma\beta\,\omega'/c + \gamma\, k^{z\prime}$$

$$k^x = k^{x\prime}$$

So

① $$k = \gamma k' + \gamma\beta k' \cos\theta'$$

② $$k\cos\theta = \gamma\beta k' + \gamma k' \cos\theta'$$

③ $$k\sin\theta = k'\sin\theta$$

Thus we see that (from ① $-\beta$② see eqn numbers

$$k(1-\beta\cos\theta) = \gamma(1-\beta^2)k'$$

$$\boxed{k = \frac{k'/\gamma}{(1-\beta\cos\theta)}} = \boxed{\frac{k_0 V/c}{(1-\beta\cos\theta)} = k}$$

This is what we wanted it shows that at angle θ the wave number is $k_0 V/(1-\beta\cos\theta)$