Radiation from a pair of Oscillators

a) This is dipole radiation. The dipole moment

$$\vec{p} = q\vec{r}_1 + q\vec{r}_2$$

$$\vec{p} = 2qd\,e^{-i\omega t}\,\hat{y}$$

Then from the usual formulas $\vec{p}(t) = \vec{p}_\omega e^{-i\omega t}$

$$\frac{d\overline{P}}{d\Omega} = \frac{1}{16\pi^2 c^3} \underbrace{\frac{|\vec{n}\times\vec{n}\times\vec{p}_\omega|^2}{2}}_{}$$ ← time ave

Now $\vec{n}\times\vec{n}\times\vec{p}_\omega = -\vec{p}_\omega + \vec{n}(\vec{n}\,p_\omega)$. But in this case $\vec{n}\cdot\vec{p}_\omega = 0$, since $\vec{p}_\omega \propto \hat{y}$ but $\vec{n}$ lies in the $x-z$ plane: Thus

$$\frac{d\overline{P}}{d\Omega} = \frac{1}{16\pi^2 c^3}\frac{(2q\,d)^2}{2}$$

b) The electric field for dipole radiation

$$\vec{E}(t,\vec{r}) = \frac{1}{4\pi r c^2}\,\vec{n}\times\vec{n}\times\ddot{\vec{p}}(t_e)$$

Now $\vec{p}(t_e) = \vec{p}_\omega \cos(-\omega(t - r/c))$

$$\ddot{\vec{p}}(t_e) = -\omega^2\vec{p}_\omega \cos(\omega t - kr)$$

So

$$\vec{E}(t,\vec{r}) = \frac{\cos(\omega t - kr)}{4\pi r c^2} \; \vec{n} \times \vec{n} \times \vec{p}_\omega \; (-\omega^2)$$

where $p_\omega = 2dq\,\hat{y}$. On the z axis we have

$$\vec{n} \times \vec{n} \times \vec{p}_\omega = \hat{z} \times \hat{z} \times \vec{p}_\omega = -\vec{p}_\omega$$

So

$$\vec{E}(t,R) = \frac{k^2}{4\pi R} \cos(\omega t - kR)\,\hat{y}\,(2qd)^2 \qquad \text{(z axis)}$$

On the y axis $\vec{n} \times \vec{n} \times \vec{p}_\omega = 0$

$$\vec{E}(t,R) = 0 \qquad \text{(y-axis)}$$

This is clear for $\vec{n}$ along the y-axis the currents $\partial_t \vec{J}$ are along the line of sight and do not drive the $\vec{E}$ & $\vec{B}$ fields which must be transverse to the line of sight.

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c) In this case

$$\vec{A}_{rad}(t,\vec{r}) = \frac{1}{4\pi r c}\int d^3r_o\, \vec{J}(T,\vec{r}_o)$$

where

$$\vec{J}_1 = q\, v_1(T)\, \delta^3(\vec{r}_o - \vec{R}_1(T))$$

For particle 1 and particle 2

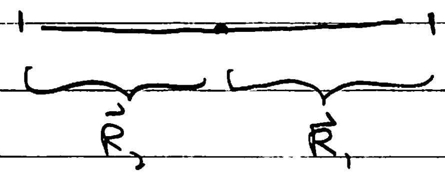
$$v_1(T) = -i\omega d\, e^{-i\omega T}\, \hat{y}$$

Here $T = t - \frac{r}{c} + \frac{n \cdot \vec{r}_o}{c}$

Now think physically

(★) $$\vec{A}_{rad}(t,\vec{r}) = \frac{\hat{y}}{4\pi r c}\, q\, (-i\omega d)\, e^{-i\omega(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{R}_1(T)}{c})} + \frac{\hat{y}}{4\pi r c}\, q\, (-i\omega d)\, e^{-i\omega(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{R}_2(T)}{c})}$$

The positions of particle 1 and particle 2 are essentially constant in time. Just look at picture



$$\vec{R}_1 = l\,\hat{x} \qquad \vec{R}_2 = -l\,\hat{x}$$

So

$$\vec{A}_{rad} = \frac{\hat{y}}{4\pi r c}\, q\,(-i\omega d)\, e^{-i\omega t + ikr} \left[e^{-i\frac{\omega}{c} l\, \vec{n}\cdot\hat{x}} + e^{+i\frac{\omega}{c} l\, \vec{n}\cdot\hat{x}} \right] \quad (1)$$

So for $\vec{n} = (\sin\theta, 0, \cos\theta)$

$$\vec{A}_{rad} = \hat{y}\, q\,(-i\omega d)\, \frac{e^{-i\omega t + ikr}}{4\pi r c}\, 2\cos(\sin\theta\, kl) \quad (2)$$

So

$$\frac{d\bar{P}}{d\Omega} = \frac{c}{2} \left| \frac{-i\omega r}{c}\, \vec{n}\times\vec{n}\times\vec{A}_{\omega,rad} \right|^2$$

$$= \frac{1}{32\pi^2 c^3}\, \omega^4\, (2qd)^2 \left[\cos(\sin\theta\, kl) \right]^2$$

So we note for $k\ell \ll 1$ we recover the result of part (a)

$$\frac{\overline{dP}}{d\Omega} \longrightarrow \frac{\omega^4}{32\pi^2 c^3}(2qd)^2$$

e) The analysis is the same but the sign of the second term is reversed leading to (see pg. 3 Eq ($\bigstar$))

$$\frac{\overline{dP}}{d\Omega} = \frac{\omega^4}{32\pi^2 c^3}(2qd)^2\left[\sin(\sin\theta\, k\ell)\right]^2$$

For $k\ell \ll 1$ we see that since $k = \omega/c$

$$\frac{\overline{dP}}{d\Omega} \propto \left(\frac{\omega}{c}\right)^6$$

This is characteristic of quadrupole radiation

A small sphere

a) $$S = \int d^4x \; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{J^\mu}{c} A_\mu$$

Varying the action $A_\mu \to A_\mu + \delta A_\mu$

$$\delta S = \int d^4x \; -\frac{1}{2} F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + \frac{J^\mu}{c} \delta A_\mu$$

Integrate by parts

$$\delta S = \int d^4x \left[+\frac{1}{2} \partial_\mu F^{\mu\nu} \delta A_\nu - \frac{1}{2} \partial_\nu F^{\mu\nu} \delta A_\mu \right] + \frac{J^\mu}{c} \delta A_\mu$$

After relabelling and using $F^{\mu\nu} = -F^{\nu\mu}$ we have

$$\delta S = \int d^4x \; \delta A_\rho \left[\partial_\sigma F^{\sigma\rho} + \frac{J^\rho}{c} \right]$$

So the EOM is

$$\boxed{-\partial_\sigma F^{\sigma\rho} = J^\rho / c}$$

This is <u>not</u> all of the maxwell equations

A small sphere pg. 2

The residual equations are $\partial_{[\mu}F_{\nu\rho]} = 0$ or

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

the Bianchi identity. This guarantees that $F_{\mu\nu}$ can be written

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

b) The two invariants are

$$F_{\mu\nu} F^{\mu\nu} = -2(E^2 - B^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\, \vec{E}\cdot\vec{B}$$

c) The covariant transformation rule is

$$\underline{F}^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma}$$

Here

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Sphere 3

Now

$$\underline{F}^{01} = L^0{}_\rho L^1{}_\sigma F^{\rho\sigma}$$

$$\underline{F}^{01} = L^0{}_1 L^1{}_1 F^{11} = 0$$

$$E^x = 0$$

$$\underline{F}^{02} = L^0{}_1 L^2{}_2 F^{12}$$

$$\underline{E}^y = -\gamma\beta \ (B^z)$$

and

$$\underline{F}^{03} = L^0{}_1 L^3{}_3 F^{13}$$

$$\underline{E}^z = -\gamma\beta \ (-B^y)$$

$$E^z = \gamma\beta B^y$$

$$\underline{F}^{12} = L^1{}_\rho L^2{}_\sigma F^{\rho\sigma}$$

$$= L^1{}_1 L^2{}_2 F^{12}$$

$$\underline{F}^{12} = \gamma F^{12}$$

$$\underline{B}^z = \gamma B^z$$

Then

$$\underline{F}^{13} = L^1{}_\rho L^3{}_\sigma F^{\rho\sigma}$$

$$\underline{B}^y = \gamma B^y$$

And

$$F^{23} = L^2{}_\rho L^3{}_\sigma F^{\rho\sigma}$$

$$F^{23} = F^{23} \qquad \underline{B}^x = B^x$$

To summarize

$$\underline{\vec{E}} = \gamma \vec{\beta} \times \vec{B}$$

$$\underline{B}_{\parallel} = B_{\parallel}$$

$$\underline{\vec{B}}_{\perp} = \gamma \vec{B}_{\perp}$$

Note

$$\underline{\vec{E}} = \vec{\beta} \times \underline{\vec{B}}$$

d) From the fact that

$$\underline{\vec{E}} = \vec{\beta} \times \underline{\vec{B}}$$

$\underline{E} \cdot \underline{B} = 0$. This follows from the invariant

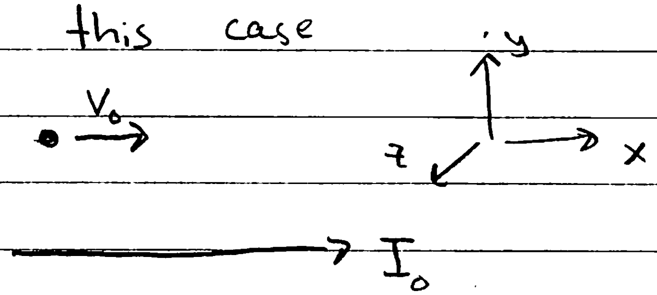
Sphere 5

If

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$$

vanishes in one frame it vanishes in all. In the original frame $\vec{E}=0$ so $\vec{E}\cdot\vec{B}$ is zero. This is true in one frame it is true in all

e) In this case



The magnetic field is in the z-direction (see coordinates above)

$$\vec{B} = \frac{I_o/c}{2\pi y}\,\hat{z} = B_z(y)\,\hat{z}$$

The E-field is with boost vector $\vec{\beta} = +\frac{V_o}{c}$

$$\vec{E} = +\frac{V_o}{c}\,\hat{x}\times B_z(y)\,\hat{z}$$

$$\vec{E} = -\frac{V_o}{c}\,B_z(y)\,\hat{y}$$

Sphere 6

So the force

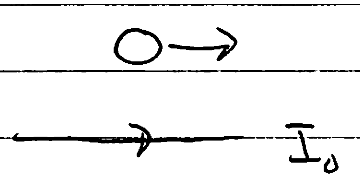
$$F^j = p^i \partial_i E^j$$

$$F^y = \alpha E^y \partial_y E^y \Big|_{y=R}$$

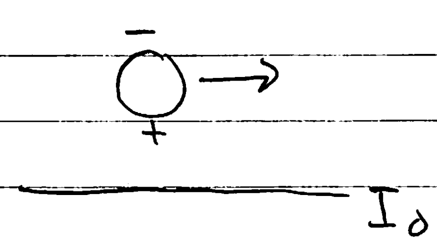
$$= -\alpha \frac{V_0}{c} \frac{I/c}{2\pi R} \frac{I/c}{2\pi} \frac{V_0}{c} \frac{\partial}{\partial y} \frac{-1}{y} \Big|_{y=R}$$

$$\boxed{F^y = -\frac{\alpha}{R^3} \frac{(I/c)^2}{(2\pi)^2} \left(\frac{V_0}{c}\right)^2} \qquad (\text{Eq} \bigstar)$$

We can understand it intuitively as follows
The charge carriers in the metal experience a force



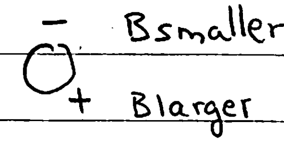
The force $q \vec{V}_0 \times \vec{B}/c$ is down for plusses and up for minus. This polarizes the the sphere



Sphere 7

The plus charges then experience a slightly larger force down

$$\vec{F} = Q \frac{\vec{v}}{c} \times \vec{B}_{larger}$$



because they are closer to the wire then the negative charges. The net force is

$$F_{net} \sim \left(Q \frac{v}{c} B_{larger} - Q \frac{v}{c} B_{smaller} \right) (-\hat{y})$$

$$\sim Q \frac{v}{c} \left(-a \frac{\partial B}{\partial y} \right) (-\hat{y})$$

We can estimate the induced charge Q. The induced charge Q is such that the electrostatic attraction balances the Lorentz force

$$\frac{Q^2}{a^2} \sim Q \frac{v}{c} B$$

$$Q \sim a^2 \frac{v}{c} B$$

So

$$F_{net} \sim a^3 \left(\frac{v}{c}\right)^2 B \left(-\frac{\partial B}{\partial y} \right) (-\hat{y})$$

This is the order of magnitude of Eq (☆) on the previous page, $\alpha \equiv 4\pi a^3$ is the polarizability.

Radiation from a kick

a) To first order the particles motion is constant

$$z = V_0 t$$

Then

$$\frac{dp^x}{dt} = F_0 \sin(k_0 v t)$$

$$\frac{dx}{dt} = \frac{c^2 p^x}{E}$$

$$\frac{dx}{dt} \simeq \frac{c^2 p^x(t)}{E_0}$$

$$\frac{d^2x}{dt^2} = \frac{c^2 \dot{p}^x(t)}{E_0} = \frac{F_0}{\gamma_0 m} \sin(k_0 v t)$$

b) Then

$$\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{|n \times (\vec{n} - \vec{\beta}) \times \vec{a}|^2}{(1 - n \cdot \beta)^5}$$

For directly forward $\vec{\beta} = \beta \vec{n}$, $1 - n \cdot \beta = 1 - \beta$

Then

$$\frac{dP}{d\Omega}(T) = \frac{q^2}{16\pi^2 c^3} \frac{|\vec{n} \times \vec{n} \times a|^2}{(1-\beta)^5} (1-\beta)^2$$

$$= \frac{q^2}{16\pi^2 c^3} \frac{1}{(1-\beta)^3} a_T^2$$

$$= \frac{q^2}{16\pi^2 c^3} (2\gamma^2)^3 \left(\frac{F_0}{\gamma m}\right)^2 \sin^2(k_0 v t)$$

$$\frac{dW}{dT\, d\Omega} = \frac{q^2}{2\pi^2 c^3} \gamma^4 \left(\frac{F_0}{m}\right)^2 \sin^2(k_0 v T)$$

Integrating over time $\int_{\text{period}} dT \sin^2(k_0 v T) = \frac{1}{2}(\text{period})$

$$\frac{dW}{d\Omega} = \frac{q^2}{2\pi^2 c^3} \gamma^4 \left(\frac{F_0}{m}\right)^2 \frac{1}{2} \frac{2\pi}{k_0 V}$$

|

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c) To find the total we use the Larmour

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^4 a_\perp^2$$

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^4 \left(\frac{F_0}{\gamma m}\right)^2 \sin^2(k_0 v T)$$

Integrating over time from

$$T = -\frac{\pi}{k_0 v} \ldots \frac{\pi}{k_0 v}$$

(This is the time that it interacts with the force)

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} \gamma^2 \left(\frac{F_0}{m}\right)^2 \frac{1}{2} \frac{2\pi}{k_0 v}$$

d) To find the frequency spectrum we use

$$\vec{E}(\omega,r) = \frac{q}{4\pi r c^2} e^{i\omega r/c} \int_{-\infty}^{\infty} dT\, e^{+i\omega(T - \frac{\vec{n}}{c}\cdot\vec{r}_*(T))}$$

$$\frac{[n \times (\vec{n} - \vec{\beta}) \times \vec{a}]}{(1 - \vec{n}\cdot\vec{\beta})^2}$$

Now, $\vec{n} = \hat{z}$ and $\vec{n} - \vec{\beta} = (1-\beta)\hat{z}$, $(1 - n\cdot\beta) = 1-\beta$

and $r_*(T) = vT$ so

$$\vec{E}(\omega,\vec{r}) = \frac{q\, e^{i\omega r/c}}{4\pi r c^2} \cdot \int_{-\infty}^{\infty} dT\, e^{i\omega(1-\beta)T}$$

$$\frac{\frac{F_0}{\gamma m} \sin(k_0 v T)\,(-\hat{x})}{(1-\beta)}$$

So

$$2\pi \frac{dW}{d\omega\, d\Omega} = c|rE(\omega,r)|^2$$

using $\left(\frac{F}{\gamma m}\frac{1}{(1-\beta)}\right)^2 = 4\gamma^2\left(\frac{F_0}{m}\right)^2$

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We have

$$\frac{2\pi\, dW}{d\omega\, d\Omega} = \frac{q^2 \gamma^2}{4\pi^2 c^3}\left(\frac{F}{m}\right)^2 |I|^2$$

Where

$$I = \int_{-\infty}^{\infty} dT\, e^{i\omega(1-\beta)T} \sin(k_0 V T)$$

Define $\xi = k_0 V T$ $\quad u = \frac{\omega(1-\beta)}{k_0 V}$

$$I = \frac{1}{k_0 V} \int_{-\pi}^{\pi} d\xi\, e^{iu\xi} \sin(\xi) \qquad (\text{Eq.} \bigstar)$$

$$= \frac{2i}{k_0 V} \frac{\sin(\pi u)}{(1-u^2)}$$

Then

$$\boxed{\frac{2\pi\, dW}{d\omega\, d\Omega} = \frac{q^2}{\pi^2 c^3}\left(\frac{F}{m}\right)^2 \frac{\gamma^2}{(k_0 V)^2}\left[\frac{\sin(\pi u)}{(1-u^2)}\right]^2 \qquad u \equiv \omega(1-\beta)/k_0 V}$$

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We can check this result by integrating over ω or u

$$\int_{-\infty}^{-\infty} \frac{d\omega}{2\pi} \ldots = \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{k_o V}{(1-\beta)} \ldots$$

Then by convolution theorem (notice Eq ★ is a fourier transfrm

$$\int_{-\infty}^{\infty} \frac{du}{2\pi} 4\left(\frac{\sin \pi u}{(1-u^2)}\right)^2 = \int_{-\pi}^{\pi} dx \sin^2 x$$

$$= \pi$$

So

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} 2\pi \frac{dW}{d\omega d\Omega}$$

$$= \frac{q^2}{4\pi^2 c^3} \left(\frac{F}{m}\right)^2 \frac{\gamma^2}{(k_o V)^2} \pi \frac{k_o V}{(1-\beta)}$$

$$\frac{1}{1-\beta} = 2\gamma^2$$

$$\frac{dW}{d\Omega} = \frac{q^2}{2\pi c^3} \left(\frac{F}{m}\right)^2 \frac{\gamma^4}{k_o V}$$

This agrees with part (b).

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e) The typical frequency is

$$\omega \sim \frac{k_o V}{(1-\beta)} \sim \gamma^2 k_o V$$

This is expected. The formation time (the time that the particle is accelerated) is of order $\sim \frac{1}{k_o V}$ (The duration of the acceleration is $2\pi / k_o V$)

Then if a wave was formed/emitted over a time ΔT, then it arrives at detector over a time Δt

$$\Delta t = \frac{\Delta T}{\frac{\Delta T}{\Delta t}} = \frac{\Delta T}{\frac{1}{(1-n\cdot\beta)}}$$

So the duration of the pulse seen by the detector is:

$$\Delta t \sim \frac{2\pi}{k_o V}(1-\beta)$$

So by the uncertainty principal, the typical frequency is

$$\omega \sim \frac{1}{\Delta t} \sim \frac{k_o V}{(1-\beta)} \sim \gamma^2 k_o V$$