Problem 1

a) We will use Larmour formulas

$$\vec{E}_+ = \frac{q}{4\pi r c^2} \vec{n} \times \vec{n} \times \vec{a}_+ (t_e) \leftarrow \text{positive}$$

negative charge

$$\vec{E}_- = \frac{-q}{4\pi r c^2} \vec{n} \times \vec{n} \times \vec{a}_- (t_e) \leftarrow \text{negative charge}$$

Here

$$\vec{r}_+(t_e) = \frac{s}{2} (\hat{x} + i\hat{y}) e^{-i\omega_0 (t - r/c)} \qquad t_e = t - r/c$$

$$\vec{a}_+ = -\omega_0^2 \left(\frac{s}{2}\right) (\hat{x} + i\hat{y}) e^{-i\omega_0 t + ik_0 r}$$

$$\vec{a}_- = +\omega_0^2 \left(\frac{s}{2}\right) (\hat{x} + i\hat{y}) e^{-i\omega_0 t + ik_0 r}$$

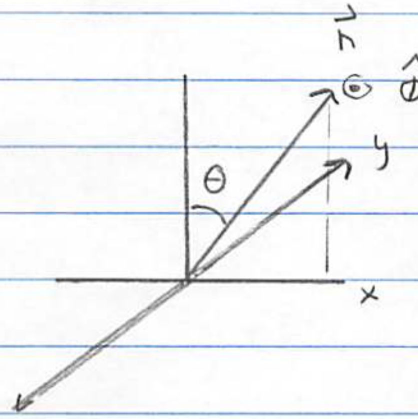
Then

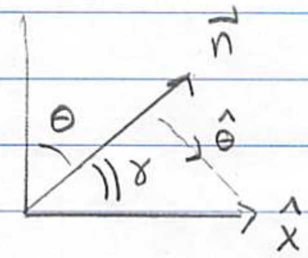
$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$\vec{E} = -\left(\frac{q s}{4\pi c^2}\right) \frac{e^{-i\omega_0 t + ik_0 r}}{r} \omega_0^2 \left[\vec{n} \times \vec{n} \times (\hat{x} + i\hat{y}) \right]$$

Sorting out $\vec{n} = (\sin\theta, 0, \cos\theta)$

$$\vec{n} \times \vec{n} \times \hat{y} = -\hat{y} = -\hat{\phi}$$





Now $\vec{n} \times \vec{n} \times \hat{x} = -\hat{x} + \vec{n}\,(\vec{n}\cdot\hat{x})$

$$\hat{x} = \cos\gamma\ \vec{n} + \sin\gamma\ \hat{\theta}$$

$$= \sin\theta\ \vec{n} + \cos\theta\ \hat{\theta}$$

$\vec{n} \times \vec{n} \times \vec{x} = -($piece of $\hat{x}$ $\perp$ to $\vec{n}$ $)$

$= -\cos\theta\ \hat{\theta}$ (see picture)

So

$$\vec{E} = +\frac{qs}{4\pi}\left(\frac{\omega_0}{c}\right)^2 \frac{e^{-i\omega_0 t + i k_0 r}}{r}\left[\cos\theta\ \hat{\theta} + i\hat{\phi}\right]$$

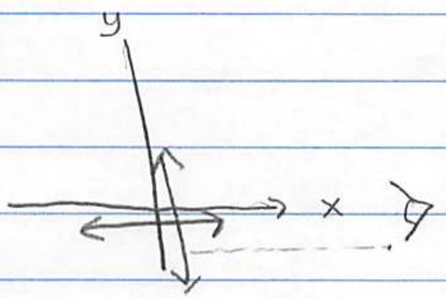
There is no $\hat{r}$ component because the radiation field is transverse to propagation in $\hat{r}$ direction

b) On the $\hat{x}$-axis, $\cos\theta = 0$ $\quad \hat{\phi} = \hat{y}$

$r = x$, Taking the real part $\text{Re}\; i e^{-i\theta} = -\sin\theta$

$$\vec{E} = +\frac{qs}{4\pi}\left(\frac{\omega_0}{c}\right)^2 \frac{\sin(\omega t - kr)}{r}\,\hat{y}$$

The polarization is y-direction. The reason for this is because: the rotational motion can be thought of as a super-position of x-oriented dipole and a y-oriented dipole;



Only the y-oriented dipole is transverse to the line of sight (the x-axis)

On the z-axis $\cos\theta = 1$ $\quad \hat{\theta} = \hat{x}$ $\quad \hat{\phi} = \hat{y}$

$$\vec{E} = \frac{qs}{4\pi}\left(\frac{\omega_0}{c}\right)^2 \left[\frac{\cos(\omega t - kr)}{r}\,\hat{x} + \frac{\sin(\omega t - kr)}{r}\,\hat{y}\right]$$

This is circular polarization. The polarization vector follows the polarization of the rotational motion

c) $$\frac{dW}{dT\,d\Omega} = \overline{c\,|r\,E(t,r)|^2} = \frac{c}{2}\,|r\,\vec{E}_\omega|^2$$

$(\hat{\phi}$ component$)^2$

$$= \frac{c\,(qs)^2}{2\cdot 16\pi^2}\,k_0^4\,\left[\cos^2\theta + 1\right]$$

$(\hat{\theta}$ component$)^2$

We used that

$$E(t,r) = E_\omega\, e^{-i\omega_0 t}$$

$$E_\omega = \frac{(q_0 s)}{4\pi}\left(\frac{\omega_0}{c}\right)^2 \frac{e^{ik_0 r}}{r}\left[\cos\theta\,\hat{\theta} + i\hat{\phi}\right]$$

d) Then on the z axis:

$$\vec{E}(t,r) = \left[\left(\frac{qs}{4\pi}\right)\left(\frac{\omega_0}{c}\right)^2 \frac{e^{-i\omega_0 t + ikr}}{r}\,(\hat{x} + i\hat{y})\right.$$

$$\left. + \text{complex conjugate}\right]/2$$

$$\vec{E}(\omega,r) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\,\vec{E}(t,r) \qquad t_e = t - r/c$$

$$= \int_{-\infty}^{\infty} dt_e\, e^{+i\omega(t_e + r/c)}\,\vec{E}(t_e,\vec{r})$$

It is easiest to use

$$\vec{E}(\omega, r) = -\frac{i\omega}{c}\, \vec{n} \times \vec{n} \times \vec{A}_{rad}(\omega, r)$$

Here

$$\vec{A}_{rad}(t, r) = \frac{q}{4\pi r c}\left(\vec{v}_+(t_e) - \vec{v}_-(t_e)\right) = \frac{2q}{4\pi r c}\, \vec{v}_+(t_e)$$

Note

$$v_+ = \frac{d}{dt}\left(\cos\omega_0 t\, \hat{x} + \sin\omega_0 t\, \hat{y}\right)\frac{s}{2}$$

So $\quad -2\vec{v}_+ = (+\omega_0 \sin\omega_0 t\, \hat{x} - \omega_0 \cos\omega_0 t\, \hat{y})\, s$

Then $\quad \vec{n} \times \vec{n} \times 2v_+(t_e) = -2\vec{v}_+(t_e) \quad$ for $\vec{n}$ on z-axis

and thus fourier transforming $\vec{A}(t)$

stops at zero because $v=0$ afterward

$$\vec{E}(\omega, r) = -\frac{i\omega}{c}\frac{q\, e^{i\omega r/c}}{4\pi r c}\int_{-\infty}^{0} dt_e\, e^{i\omega t_e}\left(\sin\omega_0 t_e\, \hat{x} - \cos\omega_0 t_e\, \hat{y}\right) \cdot (\omega_0 s)$$

We need two integrals:

$$I_1(\omega) = \int_{-\infty}^{0} dt_e\, e^{i\omega t_e}\left(\sin\omega_0 t_e\right) e^{\varepsilon t}$$

convergence factor cutting off $t \to -\infty$

And

$$I_2(\omega) = \int_{-\infty}^{0} dt_e \; e^{i\omega t_e} \cos\omega_0 t_e \; e^{\varepsilon t}$$

Writing $\sin\omega_0 t_e = \dfrac{e^{i\omega_0 t_e} - e^{-i\omega_0 t_e}}{2i}$, $\cos\omega_0 t_e = \dfrac{e^{i\omega_0 t_e} + e^{-i\omega_0 t_e}}{2}$

and using

$$\int_{-\infty}^{0} e^{+i(\omega \pm \omega_0 - i\varepsilon)t} \, dt = \frac{1}{i(\omega \pm \omega_0 - i\varepsilon)}$$

We find

$$I_1 = \frac{\omega_0}{(\omega^2 - \omega_0^2)} \qquad I_2 = \frac{\omega}{(\omega^2 - \omega_0^2)}$$

Thus

$\hat{x}$ direction $\quad \hat{y}$ direction

$$c r^2 \left| \vec{E}(\omega, r) \cdot \vec{E}^*(\omega, r) \right| = c\left(\frac{\omega}{c}\right)^2 \frac{(qs)^2}{16\pi^2} \left(\frac{\omega_0}{c}\right)^2 \left[\frac{\omega_0^2 + \omega^2}{(\omega^2 - \omega_0^2)^2}\right]$$

And thus

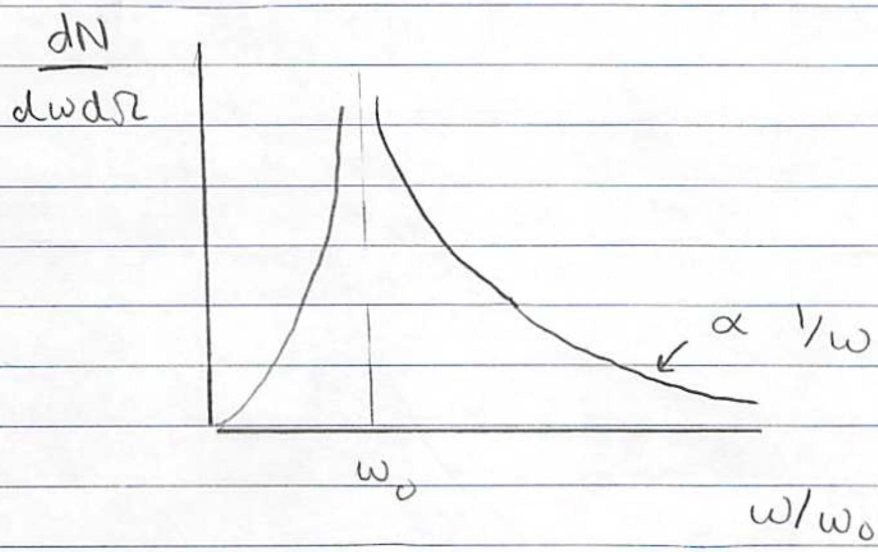
$$2\pi \frac{dW}{d\omega \, d\Omega} = \frac{c\,(qs)^2}{16\pi^2} \left(\frac{\omega}{c}\right)^2 \left(\frac{\omega_0}{c}\right)^2 \left[\frac{\omega_0^2 + \omega^2}{(\omega^2 - \omega_0^2)^2}\right]$$

Now

$$\hbar\omega \frac{dN}{d\omega\, d\Omega} = 2\left(\frac{dW}{d\omega\, d\Omega}\right)_{\omega>0}$$

Thus

$$\frac{dN}{d\omega\, d\Omega} = \left(\frac{q^2}{4\pi\hbar c}\right)\frac{(k_0 s)^2}{4\pi^2}\,\frac{1}{\omega_0}\left(\frac{\omega}{\omega_0}\right)\left(\frac{1+(\omega/\omega_0)^2}{((\omega/\omega_0)^2-1)^2}\right)$$



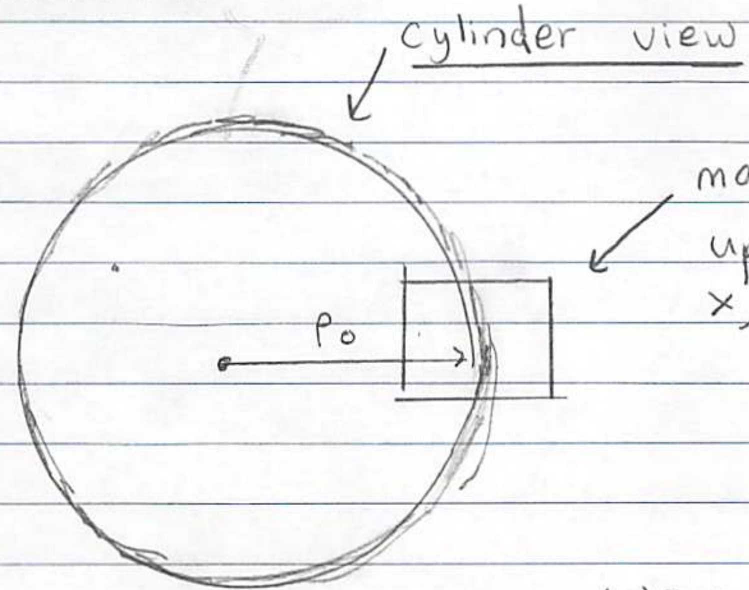
Note at high frequency one sees a characteristic bremsstrahlung tail $\propto 1/\omega$

$$\frac{dN}{d\omega\, d\Omega} = \frac{\alpha_{em}}{4\pi^2}\left(\frac{v_0}{c}\right)^2\frac{1}{\omega} \qquad \frac{v_0}{c} = s\omega_0/c = k_0 s$$

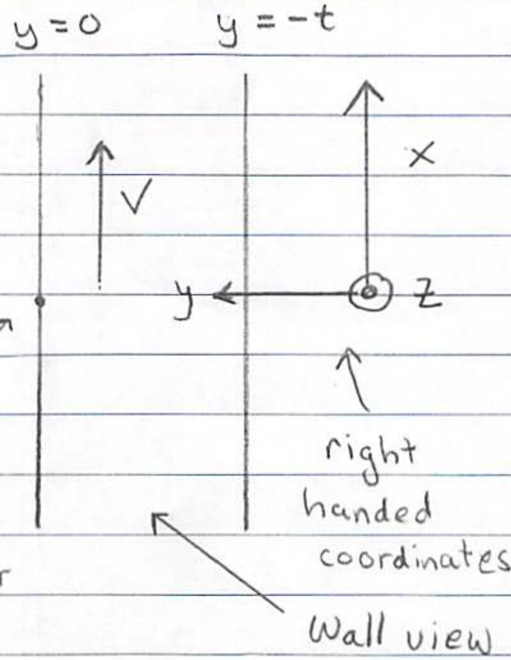
$$\alpha_{EM} = e^2/4\pi\hbar c$$

which is typical of an abrupt stop

Problem 2



magnify this and set up some local coordinates x, y, z



$\rho(y) = \rho_0 - y$

a) The vector potential

$\vec{A} = \frac{1}{2} \vec{B} \times \vec{\rho}$ in global cylinder coords

$A^x = \frac{1}{2} B \rho$ in x direction according to the cylinder walls coordinates

$\rho = \rho_0 - y$

$\phi = 0$

Now boost to the moving wall frame

$$\begin{pmatrix} \phi \\ \underline{A} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} B_0 \rho \end{pmatrix}$$

$\phi \simeq -\beta \frac{1}{2} B_0 \rho \qquad \underline{A} \simeq \frac{1}{2} B_0 \rho$

So in the moving frame see a potential

$$\underline{\Phi} = -\frac{\omega_0 \rho^2}{c} \frac{B_0}{2} \qquad \text{we used} \qquad v = \omega_0 \rho$$

$$\beta = \frac{\omega_0 \rho}{c}$$

Note that the coordinate ρ is transverse to the boost and is unchanged by the boost

b) Using the field transformation rules we find

$$\underline{E}_{\parallel} = \gamma E_{\parallel}$$

$$\vec{\underline{E}}_{\perp} = \gamma \vec{\underline{E}}_{\perp} + \gamma \vec{\beta} \times \vec{B}$$

So we set $E_{\parallel} = E_{\perp} = 0$ $\quad \gamma \simeq 1$ and find

see coordinate

(Eq★) $$\vec{\underline{E}} = \frac{\omega_0 \rho}{c} \hat{x} \times B_0 \hat{z} = \frac{\omega_0 \rho}{c} (-\hat{y})$$

Where we used the wall coordinates of part a)

To check consisteny with part a) we write

$$\rho^2 = (\rho_0 - y)^2 \simeq \rho_0^2 - 2\rho_0 y$$

(see picture for coordinates!)

Now we can check consistency:

$$E_y = -\frac{\partial}{\partial y}\Phi = -\frac{\partial}{\partial y}\left(-\frac{\omega_0 \rho^2}{c}\frac{B_0}{2}\right)$$

$$= -\frac{\partial}{\partial y}\left(-\frac{\omega_0 \rho_0^2}{2c} + \frac{\omega \rho_0 y\, B_0}{c}\right)$$

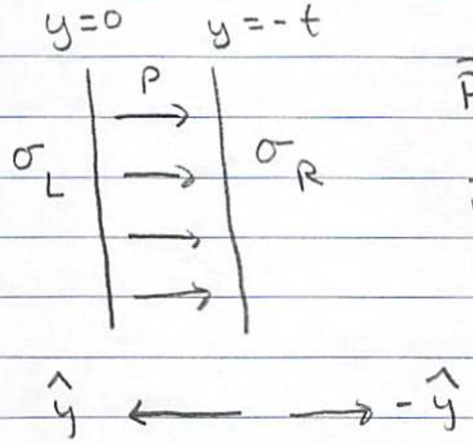
$$= -\frac{\omega \rho_0 B_0}{c}$$

This agrees w/ Eq ★ to zeroth order in y where $\rho \simeq \rho_0$

i.e.

$$\vec{E} = -\frac{\omega \rho_0 B_0}{c}\,\hat{y}$$ (see coordinates on page 1)

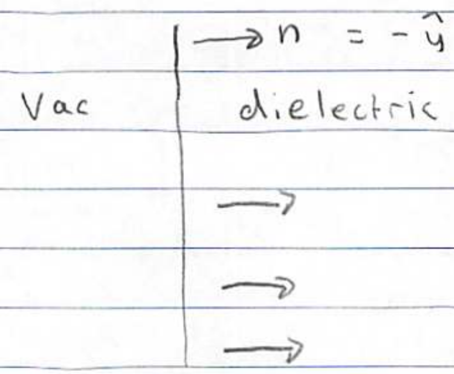
c)



$$\vec{P} = \chi \vec{E}$$

$$\vec{P} = \chi \frac{\omega_0 \rho_0 B_0}{c}(-\hat{y})$$

Now we can use the boundary conditions to find the surface charge



$$\sigma_L = -\vec{n}\cdot(\vec{P}_2 - \vec{P}_1)$$

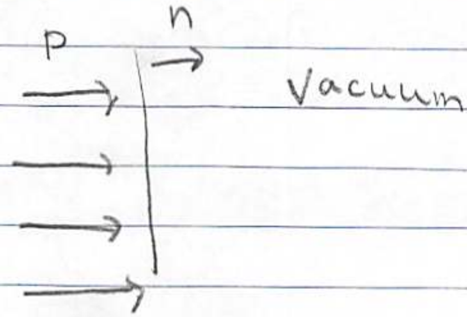
$$= -\vec{n}\cdot\vec{P}_2$$

$$\sigma_L = -\chi \omega_0 \rho_0 B_0$$

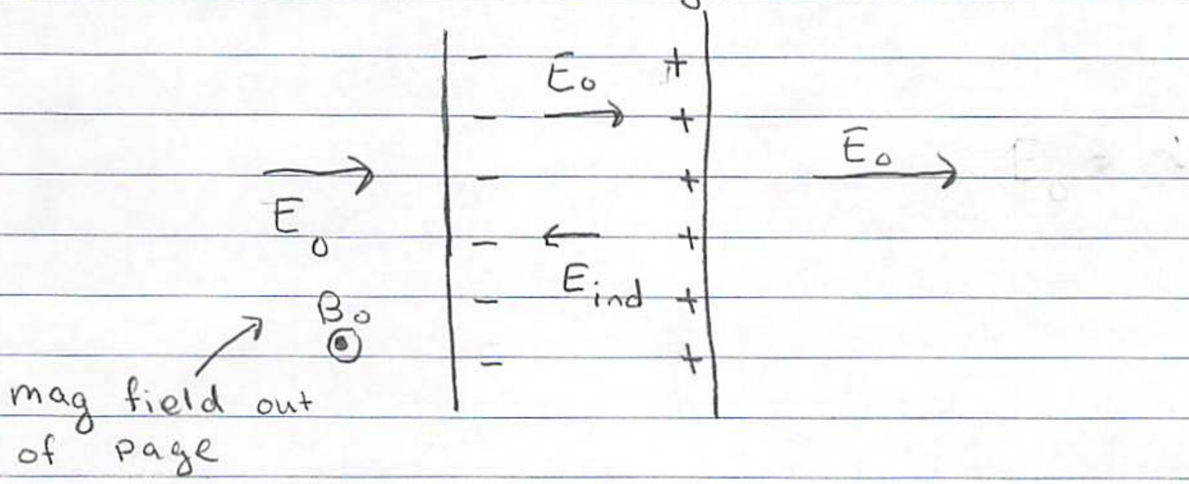
Similarly for σ_R

$$\sigma_R = -\vec{n} \cdot (\cancel{\vec{P}_2} - \vec{P}_1)$$

$$\sigma_R = + \chi \omega_0 \rho_0 B_0$$



So the overall picture is one of a parallel plate capacitor in an electric and magnetic field.



Here $E_o = \frac{\omega_o \rho_o}{c} B_o$. The induced surface charge is $\sigma_R = \chi \omega_o \frac{\rho_o}{c} B_o$ induces an electric field

$$\vec{E}_{ind} = \sigma_R \, \hat{y}$$

d) The charge is unchanged by a boost back to the "Lab" frame

Boost in negative x-direction

Since

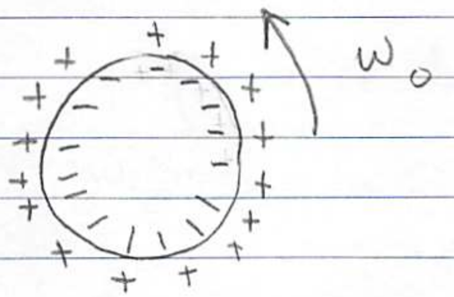
$$\begin{pmatrix} \rho_{lab} \\ J^x_{lab} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

$$\rho_{Lab} \simeq \rho_{wall}$$

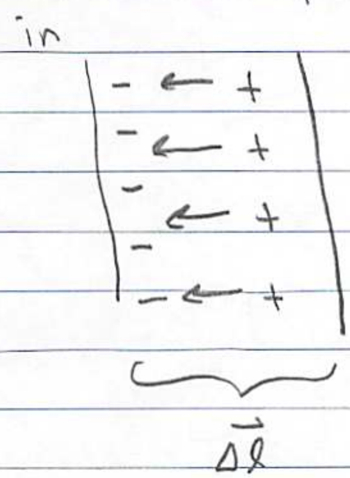
$$J_{Lab} \simeq \rho_{wall} V_{wall}$$

$$J^x_{Lab} = \rho_{wall}\, \omega_0 \rho_0$$

Thus in the lab frame we see the following with charge per area $\sigma_{out} = \frac{x \omega_0 \rho_0 B_0}{c}$



The electric potential difference is



$$V_{out} - V_{in} = -\vec{E} \cdot \vec{\Delta l}$$

$$\Delta V = \frac{x \omega_0 \rho_0 B_0}{c}\, t$$

thick ness

Problem 3

a) $$S = \int d\lambda \, mc \left(-\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}\right)^{1/2} + \overbrace{\frac{q}{c}\int d\lambda \frac{dx^\mu}{d\lambda} A_\mu}^{S_{int}}$$

Then under gauge-transformation

$$A_\mu \rightarrow A_\mu - \partial_\mu \Lambda \equiv A'_\mu$$

Now

$$S'_{int} = \frac{q}{c}\int d\lambda \frac{dx^\mu}{d\lambda}\left(A_\mu - \partial_\mu \Lambda\right)$$

Then the gauge term is a total derivative

$$-\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}\Lambda(x) = -\frac{d\Lambda(x(\lambda))}{d\lambda}$$

which contributes nothing as Λ vanishes at $\lambda = \pm\infty$

$$S'_{int} = -\frac{q}{c}\left[\cancelto{0}{\Lambda(\infty)} - \cancelto{0}{\Lambda(-\infty)}\right]$$

$$+ \frac{q}{c}\int d\lambda \frac{dx^\mu}{d\lambda} A_\mu = S_{int}$$

i.e. $S'_{int} = S_{int}$ implying gauge invariance!

Now

$$S = \overbrace{-\int d\lambda \; mc\,(-\dot{x}\cdot\dot{x})^{1/2}}^{S_0} + \overbrace{\frac{q}{c}\int d\lambda\; \dot{x}\cdot A}^{S_{int}}$$

We vary the action $x^\mu \to x^\mu + \delta x^\mu$

$$\delta S_0 = \int d\lambda \; \frac{+mc\,\dot{x}^\mu \frac{d}{d\lambda}\delta x_\mu}{(-\dot{x}\cdot\dot{x})^{1/2}} \qquad \dot{x}\cdot\dot{x} \equiv \frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}$$

Integrating by parts

$$\delta S_0 = \int d\lambda \left[-\frac{d}{d\lambda}\frac{m\dot{x}^\mu}{(-\dot{x}\cdot\dot{x})^{1/2}}\right]\delta x_\mu$$

Now vary the interaction

$$\delta S_{int} = \underline{\frac{q}{c}\int d\lambda \left(\frac{d}{d\lambda}\delta x_\mu\right)A_\mu} + \dot{x}^\nu \frac{\partial A_\nu}{\partial x^\mu}\delta x^\mu$$

Integrating the underlined term by parts:

$$\underline{\qquad} = \frac{q}{c}\int d\lambda\;\delta x_\mu \left(-\frac{dA^\mu}{d\lambda}\right)$$

$$= -\frac{q}{c}\int d\lambda\;\delta x_\mu \frac{\partial A^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}$$

Collecting terns and relabelling $\alpha \to \nu$

$$\delta S_0 + \delta S_{int} = \int d\lambda \, \delta x_\mu \left[-\frac{d}{d\lambda}\left(\frac{m \dot{x}^\mu}{(-\dot{x}\cdot\dot{x})^{1/2}} \right) + \right.$$

$$\left. + \frac{q}{c}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right) \frac{dx^\nu}{d\lambda} \right]$$

Choosing $\lambda = \tau$, $-\dot{x}\cdot\dot{x} = 1$, $\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\tau} = u^\mu$, we find:

$$\delta S = \int d\lambda \, \delta x_\mu \left[-\frac{d}{d\tau}(m u^\mu) + \frac{q}{c} F^\mu{}_\nu \frac{dx^\nu}{d\tau} \right] = 0$$

Or

$$\boxed{-\frac{d(m u^\mu)}{d\tau} + \frac{q}{c} F^\mu{}_\nu \, u^\nu = 0} \qquad (\text{Eq} \bigstar)$$

Then multiplying (Eq ★) by u_μ:

$$-u_\mu \frac{d(m u^\mu)}{d\tau} + \frac{q}{c} \cancelto{0}{u_\mu F^\mu{}_\nu u^\nu} = 0$$

This $F^{\mu\nu} u_\mu u_\nu = 0$ because $F^{\mu\nu}$ is antisymmetric while $u_\mu u_\nu$ is symmetric.

Thus

$$u_\mu \frac{d\, m u^\mu}{d\tau} = 0 \qquad \text{or} \qquad \frac{d}{d\tau}\left(m u_\mu u^\mu \right) = 0$$

i.e.

$$\underline{\underline{u_\mu u^\mu = \text{const}}}$$

c) From

$$m \frac{du^\mu}{d\tau} = q F^\mu{}_\nu u^\nu$$

We find (since only $F^i{}_0 \neq 0$) that

$$m \frac{du^i}{d\tau} = q F^i{}_0 u^0$$

Writing $u^i = \sinh y$ $\quad u^0 = \cosh y$ we have

$$m \cosh y \frac{dy}{d\tau} = qE \cosh y$$

$$\frac{dy}{d\tau} = \frac{qE}{m}$$

$$y = \frac{qE\tau}{m} + \text{const}$$

(page 4,5)

Then we know that at $\tau = 0$
the particle has rapidity $-y_0$

i.e. the particle moves to the left with magnitude of rapidity y_0 and is just enterring the field

Thus

$$y = \frac{qE\tau}{m} - y_0$$

And

$$u(\tau) = \sinh\left(\frac{qE\tau}{m} - y_0\right) \qquad u^0 = \cosh\left(\frac{qE\tau}{m} - y_0\right)$$

Integrating to find the position:

$$u(\tau) = \frac{dx}{d\tau} = \sinh\left(\frac{qE}{m}\tau - y_0\right)$$

$$x = \int d\tau \, \sinh\left(\frac{qE\tau}{m} - y_0\right)$$

$$= \frac{1}{(qE/m)} \cosh\left(\frac{qE\tau}{m} - y_0\right) + \text{const}$$

Then

The integration constant is chosen so that at $\tau = 0$, $x = 0$

(1) $$x(\tau) = \frac{m}{qE}\left[\cosh\left(\frac{qE}{m}\tau - y_0\right) - \cosh y_0\right]$$

Similarly

$$\frac{dt}{d\tau} = \cosh\left(\frac{qE}{m}\tau - y_0\right)$$

$$t = \int d\tau \cosh\left(\frac{qE}{m}\tau - y_0\right)$$

$$t = \frac{m}{qE}\sinh\left(\frac{qE\tau}{m} - y_0\right) + \text{const}$$

adjust constant so $t=0$ at $\tau=0$

(2) $$t = \frac{m}{qE}\left[\sinh\left(\frac{qE\tau}{m} - y_0\right) + \sinh y_0\right]$$

Then from the velocity we see that the velocity goes to zero for $\tau = \frac{m y_0}{qE}$

The time at this point is

(3) $$t = \frac{m}{qE}\sinh y_0$$

The distance is

$$x = -d_{max} = -\frac{m}{qE}\left[\cosh y_0 - 1\right]$$

The total time in the field is twice equation (3)

$$\begin{matrix}\text{time} \\ \text{in field}\end{matrix} = \frac{2m \sinh y_0}{qE}$$

Not part of exam

To find the energy loss we use the Larmour formula

$$\frac{dW}{dT} = \frac{e^2}{4\pi}\frac{2}{3}\frac{A^\mu}{c3} A_\mu$$

Where $A^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}$

$$A^x = \frac{du^x}{d\tau} = \frac{qE}{m} u^0(\tau) = \frac{qE}{m}\cosh\left(\frac{qE\tau}{m} - y_0\right)$$

$$A^0 \quad \frac{du^0}{d\tau} = \frac{qE}{m} u(\tau) = \frac{qE}{m}\sinh\left(\frac{qE\tau}{m} - y_0\right)$$

$$A^\mu A_\mu = \left(\frac{qE}{m}\right)^2\left[-\sinh^2 + \cosh^2\right] = \left(\frac{qE}{m}\right)^2$$

Then

$$W = \int dT \, \frac{e^2}{4\pi}\frac{2}{3}\left(\frac{qE}{m}\right)^2$$

Changhing variables from T to $\tau \Rightarrow dT = \gamma \, d\tau$

or

Not part of exam

$$W = \int d\tau \cosh\left(\frac{qE}{m}\tau - y_0\right)\left(\frac{qE}{m}\right)^2 \frac{e^2}{4\pi}\cdot\frac{2}{3}$$

Changing variables to $\Delta y \equiv \frac{qE}{m}\tau$ and integrating from $\Delta y = 0$ (entering the field) until $\Delta y = y_0$ (fully stopped) and multiplying by two to account for the return trip

$$W = 2\int_0^{y_0} d(\Delta y)\cosh(\Delta y - y_0)\left(\frac{e^2}{4\pi}\frac{2}{3}\left(\frac{qE}{m}\right)\right)$$

$$\underline{\underline{W = \frac{e^2}{4\pi}\frac{4}{3}\frac{qE}{m}\sinh y_0}}$$

Restoring units:

$$W = \left(\frac{e^2}{4\pi mc^2}\right) qE \;\frac{4}{3}\gamma_0 (v_0/c)$$

this is very small in practice

Note

$\frac{e^2}{4\pi mc^2}$ = classical electron radius = 2.6 fm = 2.6×10^{-15} m

So

$$\Delta \equiv \frac{W}{\gamma_o mc^2} = \left(\frac{e^2}{4\pi mc^2}\right)\left(\frac{qE}{mc^2}\right)\frac{4}{3}\left(\frac{v_o}{c}\right)$$

for $v_o/c \simeq 1$, $qE = 10^6 eV/m$ a strong lab field

and $mc^2 = 0.511 \; 10^6$ eV

$$\Delta = \frac{W}{\gamma_o mc^2} \sim 10^{-15}\left(\frac{E}{10^6 V/m}\right)\left(\frac{0.511 \times 10^6 eV}{mc^2}\right)^2$$