Helmholtz Theorems:

① If $\vec{\nabla}\cdot\vec{C} = 0$, then there exsists $\vec{D}$ such that:

$$\vec{C} = \vec{\nabla}\times\vec{D}.$$

② If $\nabla\times\vec{C} = 0$, then there exsists a scalar field S such that:

$$\vec{C} = -\vec{\nabla} S,$$

I wont prove it (but see homework) but I will show the converse, i.e.

① $\nabla\cdot(\vec{\nabla}\times\vec{D}) = 0$ and ② $\nabla\times(\vec{\nabla}S) = 0$

Prf.

① $$\partial_i C^i = \partial_i \underbrace{\varepsilon^{ijk}\partial_j D_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_i\partial_j D_k = 0$$

Because $\varepsilon^{ijk} = -\varepsilon^{jki}$ is antisymmetric while $\partial_i\partial_j = \partial_j\partial_i$ is symmetric, $\partial_x\partial_y - \partial_y\partial_x = 0$.

② Similarly, we show $\nabla\times\vec{\nabla}S = 0$

$$\underbrace{\varepsilon^{ijk}\partial_j C_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_j\partial_k S = 0$$

These are statements of differential forms $ddD = 0$

Maxwell Equations & The Helmholtz Theorems

The maxwell equations + Helmholtz theorems lead to two very important results:

I. Current Conservation
II. Gauge Potentials

First we write the MEqs. again

with source (currents)
$$\begin{cases} \nabla \cdot E = \rho \\ \nabla \times B = \vec{j}/c + 1/c\, \partial_t \vec{E} \end{cases}$$

without source
$$\begin{cases} \nabla \cdot B = 0 \\ -\nabla \times E = \frac{1}{c} \partial_t \vec{B} \end{cases}$$

I. Current Conservation.

Take the time derivative of the first equation, $\partial_t \nabla \cdot E = \partial_t \rho$, and the divergence (times c) of the second

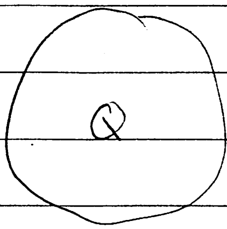
$$c\nabla \cdot (\nabla \times B) = 0 = \nabla \cdot j + \nabla \cdot \partial_t \vec{E}$$

Adding these two results

$$\star \quad \boxed{\partial_t \rho + \nabla \cdot \vec{j} = 0} \Leftarrow \text{conservation law}$$

Thus we see that maxwell equations are only consistent if charge is conserved. The conservation law imply's charge conservation

$$\partial_t Q = \int_{\text{Volume}} \partial_t \rho \, dV = \int_V -\nabla \cdot \vec{j} \, dV$$



$$= -\int_{\text{Surface}} \vec{j} \cdot d\vec{S} \longrightarrow 0$$ if the surface is taken far away

II. Gauge Potentials

Now consider the source-free equations.
(In the previous case we studied the sourced eqs.)

We can "trivially" solve these two eqs. using Helmholtz. From the third maxwell eqs

$$\nabla \cdot B = 0, \quad \text{so} \quad \boxed{B = \nabla \times \vec{A}}$$

where $\vec{A}$ is known as the vector potential.
Similarly. From the fourth Maxwell Equation

$$-\nabla \times E = \frac{1}{c} \partial_t \vec{B}$$

we find

$$-\nabla \times E = \frac{1}{c}\partial_t(\nabla \times \vec{A})$$

$$-\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

Thus we can write $\vec{E} + \frac{1}{c}\partial_t \vec{A}$ as a gradient

$$\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\vec{\nabla}\phi \Leftarrow \text{the scalar potential (voltage)}$$

or

$$\boxed{\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi}$$

From basically now on we will solve for $(\phi, \vec{A})$ instead of $(\vec{E}, \vec{B})$, since working with these variables we automatically satisfy two of the four maxwell eqs