Problem 1. Propper acceleration

A particle of mass m, starting at rest at time $t = 0$ and $x = 0$ in the lab frame, experiences a constant acceleration, a, in the x-direction in its own rest frame.

(a) The acceleration four vector

$$A^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} \tag{1}$$

is specified by the problem statement. What are the four components of the acceleration four vector in the rest frame of the particle and in the lab frame. What is the acceleration, d^2x/dt^2, in the lab frame.

(b) Show that the position of the particle as a function of time can be parameterized by a real number p

$$x = \frac{c^2}{a}\left[\cosh(p) - 1\right] \tag{2}$$

where p is related to the time t through the equation:

$$c\,t = \frac{c^2}{a}\sinh(p) \tag{3}$$

(c) Show that the parameter p is proportional proper time, $p = \frac{a}{c}\tau$.

(d) The rapidity of a particle, Y, is defined by its velocity

$$\frac{v}{c} \equiv \tanh(Y) \tag{4}$$

where $v = dx/dt$. Show that the four velocity $u^\mu = dx^\mu/d\tau$ is related to the rapidity through the hyperbolic relations.

$$(u^0/c, u^1/c) = (\cosh(Y), \sinh(Y)) \tag{5}$$

(e) Show that $Y = a\tau/c$

Remark: We see that the rapidity of the particle increases linearly with proper time during proper acceleration.

(f) If the particle has a constant decay rate in its own frame of Γ, show that the probability that the particle survives at late time t is approximately

$$\left(\frac{2at}{c}\right)^{-\Gamma c/a}$$

Problem 2. Fields from moving particle

The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$ were given in class. Choose the axes so that the charge moves along the $z-$axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define a transverse vector (or impact parameter) $\boldsymbol{b}_\perp = (x, y)$, with components x and y. Consider the fields and the source in the limit $\beta \to 1$

(a) First (keeping β finite) find the vector potential A^μ associated with the moving particle using a Lorentz transformation. Determine the field strengh tensor $F^{\mu\nu}$ by differentiating A^μ, and verify that you get the same answer as we got in class.

(b) As the charge q passes by a charge e at impact parameter $\boldsymbol{b}$, show that the accumulated transverse momentum transfer (transverse impulse) to the charge e during the passage of q is

$$\Delta \boldsymbol{p}_\perp = \frac{eq}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2 c} \tag{6}$$

(c) Show that the time integral of the absolute value of the longitudinal force to a charge e at rest at an impact parameter $b_\perp$ is

$$\frac{eq}{2\pi\gamma b_\perp c} \tag{7}$$

and hence approaches zero as $\beta \to 1$.

(d) Show that the fields of charge q can be written for $\beta \to 1$ as

$$\boldsymbol{E} = \frac{q}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z)\,, \qquad \boldsymbol{B} = \frac{q}{2\pi} \frac{\hat{\mathbf{v}}/c \times \boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z)\,. \tag{8}$$

(e) Show by explicit substitution into the Maxwell equations that these fields are consistent with the 4$-$vector source density

$$J^\alpha = q v^\alpha \delta^2(\boldsymbol{b}_\perp) \delta(ct - z) \tag{9}$$

where $v^\alpha = (c, \hat{\mathbf{v}})$.

Problem 3. Moving conductors

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor *at rest* in an electric ($\boldsymbol{E}$) and magnetic field ($\boldsymbol{B}$) the constitutive relation in a linear response approximation is known as Ohm's Law:

$$\boldsymbol{J} = \sigma \boldsymbol{E} \,. \tag{10}$$

The conductor is uncharged in its rest frame, but has a non-zero charge density in other frames.

(a) By making a Lorentz transformation of the current and fields for small boost velocities:

(i) Deduce the familiar constitutive relation[1] for a normal conductor *moving* non-relativistically with velocity $\mathbf{v}$ in an electric and magnetic field from the rest frame constitutive relation, Eq. (10). Iterpret the result in terms of the lorentz force.

(ii) Show that the charge density in the moving conductor is $\rho \simeq \mathbf{v} \cdot \boldsymbol{J}/c$. Under what conditions is the charge density positive or negative? Does a loop of wire, which in its rest frame is uncharged and carries a current I, remain uncharged when it is moving with velocity $\mathbf{v}$? Explain.

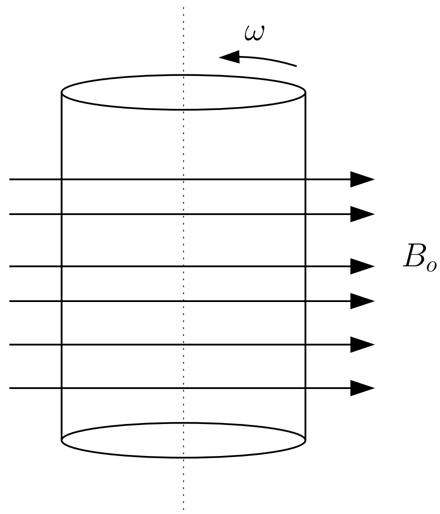
(b) In a general Lorentz frame the conductor moves with four velocity U^μ (here $U^\mu = (c, \mathbf{0})$ in the conductors rest frame, and $U^\mu = (\gamma c, \gamma \mathbf{v})$ in other frames). The constitutive relation in Eq. (10) can be expressed covariantly as

$$J^\mu = \frac{\sigma}{c} F^{\mu\nu} u_\nu \tag{11}$$

(i) Check that Eq. (11) reproduces the current and charge density of part (a) in the small velocity limit, $v \ll c$.

(c) Now consider a solid conducting cylinder of radius R and conductivity σ rotating rather slowly with constant angular velocity ω in a uniform magnetic field B_o perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder and sketch the result.

[1] $\boldsymbol{J} = \sigma(\boldsymbol{E} + \mathbf{v}/c \times \boldsymbol{B})$



(d) Determine the torque required to maintain the cylinder's constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.

(e) (**optional**) Evaluate the current numerically (in Amps) for a typical strong laboratory field $\sim 1T$, and rotation frequency $\sim 1\,\text{Hz}$, for Cu wire of radius $\sim 1\,\text{cm}$.

Problem 4. The stress tensor from the equations of motion

In class we wrote down energy and momentum conservation in the form

$$\frac{\partial \Theta^{\mu\nu}_{\rm mech}}{\partial x^\mu} = F^{\nu}_{\ \rho} \frac{J^\rho}{c} \tag{12}$$

Where the $\nu = 0$ component of this equation reflects the work done by the E&M field on the mechanical constituents, and the spatial components ($\nu = 1, 2, 3$) of this equation reflect the force by the E&M field on the mechanical constituents.

(a) Verify that

$$F^{\nu}_{\ \rho} \frac{J^\rho}{c} = \begin{cases} \boldsymbol{J}/c \cdot \boldsymbol{E} & \nu = 0 \\ \rho E^j + (\boldsymbol{J}/c \times \boldsymbol{B})^j & \nu = j \end{cases} \tag{13}$$

(b) (**Optional**) Working within the limitations of magnetostatics where

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{14}$$

show that the magnetic force can be written as the divergence of the magnetic stress tensor, $T^{ij}_B = -B^i B^j + \frac{1}{2}\delta^{ij} B^2$:

$$(\frac{\boldsymbol{J}}{c} \times \boldsymbol{B})^j = -\partial_i T^{ij}_B \tag{15}$$

(c) Consider a solenoid of infinite length carrying current I with n turns per length, what is the force per area on the sides of the solenoid.

(d) Using the equations of motion in covariant form

$$-\partial_\mu F^{\mu\rho} = \frac{J^\rho}{c} \tag{16}$$

and the Bianchi Identity

$$\partial_\mu F_{\sigma\rho} + \partial_\sigma F_{\rho\mu} + \partial_\rho F_{\mu\sigma} = 0 \tag{17}$$

show that

$$F^{\nu}_{\ \rho} \frac{J^\rho}{c} = -\frac{\partial}{\partial x^\mu} \Theta^{\mu\nu}_{\rm em} \tag{18}$$

where

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\rho} F^{\nu}_{\ \rho} + g^{\mu\nu} \left(-\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right) \tag{19}$$

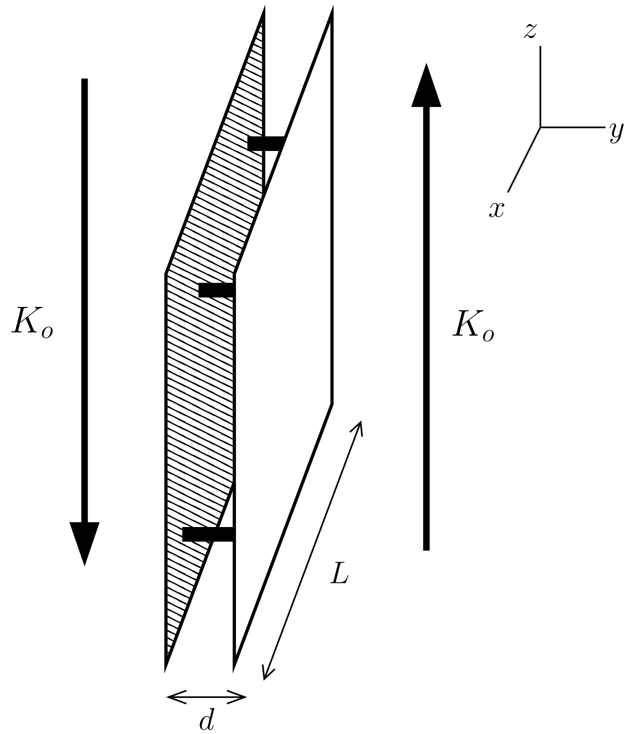
Hint: use the fact that $F^{\mu\rho}$ is anti-symmetric under interchange of μ and ρ.

(e) (**Optional**) Verify by direct substitution, using $F^{ij} = \epsilon^{ijk} B_k$, that if there is no electric field that

$$\Theta^{ij} = T^{ij}_B \,. \tag{20}$$

Problem 5. Two current sheets under Lorentz boosts

Consider two large square sheets of conducting material (with sides of length L separated by a distance d, $d \ll L$) each carrying a uniform surface current of magnitude K_o. (The total current in each sheet is $I_o = K_o L$.) The current flows up the right sheet and returns down the left sheet. The mass of the sheets is negligible. The sheets are mechanically supported by four electrically neutral columns of mass $M_{\rm col}$ and cross sectional area $A_{\rm col}$ (three shown). Neglect all fringing fields.



(a) (3 points) Write down the electromagnetic stress tensor $\Theta^{\mu\nu}_{\rm em}$ covariantly in terms of $F^{\mu\nu}$ and compute all non-vanishing components of $F^{\mu\nu}$ and $\Theta^{\mu\nu}_{\rm em}$ inside and outside of the sheets.

(b) (1 point) Compute the total rest energy of the system (or $M_{\rm tot}c^2$) including the contribution from the electromagnetic energy.

(c) (3 points) Determine the electromagnetic force per area on the current sheets (magnitude and direction) and the components of the mechanical stress tensor in the columns, $\Theta^{00}_{\rm mech}$ and $\Theta^{yy}_{\rm mech}$ (use the coordinates system in the figure). You can assume that the stress is constant across the cross sectional area of the columns.

(d) (6 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/c$ up the z-axis.

   (i) Determine the electric and magnetic fields (magnitudes and directions) using a Lorentz transformation. Check that direction of the Poynting vector measured by this observer is consistent with physical intuition.

(ii) Determine the charge and current densities in the sheets according to this observer. Are your charges and currents consistent with the fields computed in the first part of (d)? Explain.

(e) (7 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/z$ to the *right* along the y-axis (use the coordinate system shown in the figure).

(i) Determine the total mechanical energy in the columns according to this observer.

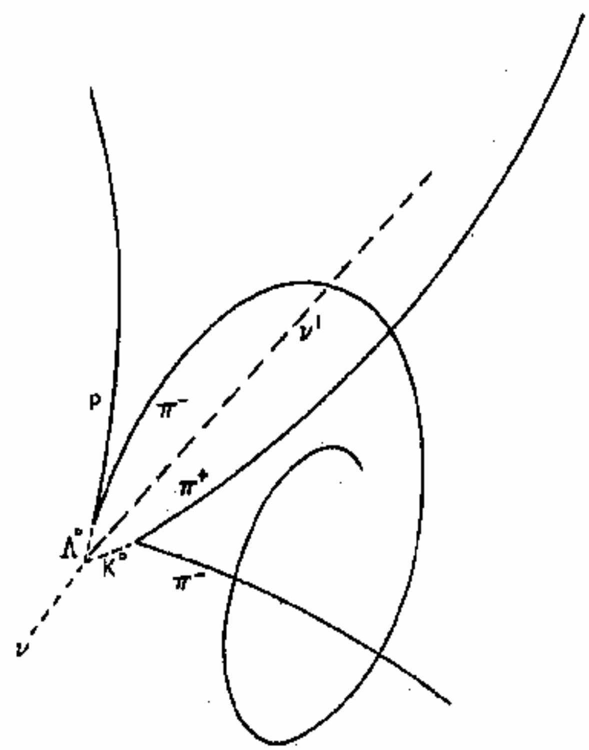
(ii) Determine the total electromagnetic energy according to this observer.

(iii) Determine the total energy of this configuration. Are your results consistent with part (b)? Explain.

Comment: There is of course stress in the sheets. But, since it does not have a yy component the stress in the sheets can be neglected in this problem.

Problem 6. (Extra-Credit) Kinematics of the Lambda decays

The lambda particle (Λ) is a neutral baryon of mass $M = 1115\,\text{MeV}$ that decays with a lifetime of $\tau = 2.9 \times 10^{-10}\,\text{s}$ into a nucleon of mass $m_1 = 939\,\text{MeV}$ and a π-meson of mass $m_2 = 140\,\text{MeV}$. It was first observed by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. In the clould chamber (and in detectors today) the charge tracks seem to appear out of nowhere from a single point (since the lambda is neutral) and have the appearance of the letter vee. Hence this decay is known as a vee decay. The particles' identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber. (In this problem M, m_1, m_2 etc are short for Mc^2, m_1c^2, m_2c^2 etc., and p_1 and p_2 are short for cp_1 and cp_2) A picture of the vee decay is shown below



(a) Using conservation of momentum and energy and the invariance of scalar products of four vectors show that, if the opening angle θ between the two tracks is measured, the mass of the decaying particle can be found from the formula

$$M^2 = m_1^2 + m_2^2 + 2\left(E_1E_2 - p_1p_2\cos\theta\right)$$

(b) A lambda particle is created with total energy of 10 GeV in and moves along the x-axis. How far on the average will it travel in the chamber before decaying? (Answer: 0.78 m)

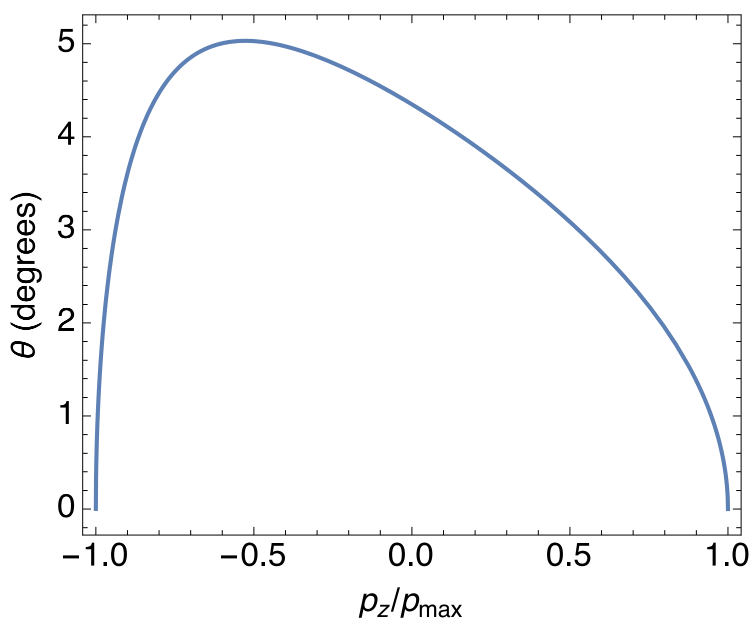
(c) Show that the momentum of the pion (or the proton) in the rest frame of the Lambda is

$$p_1 = p_2 = \sqrt{\frac{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4M^2}} \tag{21}$$

and evaluate the velocity/c of the pion v_π/c numerically. (Answer: 0.573)

Use this to determine if a pion emitted in the negative x direction in the frame of the decaying 10 GeV lambda will move forward (positive-x) or backwards (negative-x) in the lab frame.

(d) What range of opening angles will occur for a 10 GeV lambda if the decay is more or less isotropic in the lambda's rest frame? (Hint: write a program in any language (e.g. in mathematica) to plot θ vs. (p_z in the rest frame). Or you can muck about with algebra and learn less. I find $\theta = 0 \ldots 5.03^o$)



Problem 7. (Extra Credit) Kinematics of a Relativistic Rod

Consider a rod of rest length D_o. According to an inertial frame K' the rod is aligned along the x'-axis, and moves moves with velocity u' along the y' axis. The frame K' is moving to the right with velocity v relative to K in the x direction. The coordinate origins of the K and K' systems are chose so that the midpoint of the rod crosses the spatial origin at time $t = t' = 0$, *i.e.* that space-time location of the rod center intersects $t = t' = x = x' = y = y' = 0$.

(a) Find the space-time trajectory of the endpoints of the rod in frame K.

(b) At $t = 0$ in frame K, Show that the angle of the rod to the x-axis is

$$\phi = -\text{atan}(\gamma_v v u'/c^2) \tag{22}$$

where $\gamma_v = 1/\sqrt{1-(v/c)^2}$

(c) Show that the length of the rod in frame K is

$$\sqrt{\left(\frac{D_o}{\gamma}\right)^2 + \left(\frac{vu'}{c^2}\right)^2 D_o^2}$$

(d) In frame K, is the velocity of the rod $\boldsymbol{v}$ perpendicular to to its length vector $\boldsymbol{L}$. Here $\boldsymbol{L}$ points from one end of the rod to another at a given instant in time in frame K.