Problem 1. Lienard-Wiechert for constant velocity

(a) For a particle moving with constant velocity v along the $x-$axis show using Lorentz transformation that gauge potential from a point particle is

$$A^x(t, x, \boldsymbol{x}_\perp = \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + \gamma^2(x - vt)^2}} \tag{1}$$

at the point $(t, \boldsymbol{r}) = (t, x, y, z) = (t, x, \boldsymbol{b})$. So at the point $(t, 0, b, 0)$ the gauge potential A^x is is

$$A^x(t, x, y = b) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + (\gamma vt)^2}} \tag{2}$$

(b) Start by noting the definitions

$$T \equiv t - \frac{R}{c} \qquad R = |\boldsymbol{r} - \boldsymbol{r}_*(T)| \qquad \boldsymbol{R} \equiv R\boldsymbol{n} \equiv \boldsymbol{r} - \boldsymbol{r}_*(T) \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \tag{3}$$

and drawing a picture for yourself. Then, after setting $c = 1$ and $v = \beta$ to simplify algebra, show that the Lienard Wiechert result,

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{\mathbf{v}/c}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{4}$$

gives the same result as Eq. (2).

(c) Show that the Lienard-Wiechert potential, Eq. (4), and analogous equation for φ can be written covariantly

$$A^\mu(X) = -\frac{e}{4\pi} \left[\frac{U^\mu}{U \cdot \Delta X} \right]_{\rm ret} , \tag{5}$$

where ΔX^μ is the difference in the space-time coordinate four vectors of the emission and observation points, and U^μ is the four velocity of the particle. What is $\Delta X \cdot \Delta X \equiv \Delta X^\mu \Delta X_\mu$? Can $[]_{\rm ret}$ be expressed covariantly?

Lienard Wiechert

a) Note

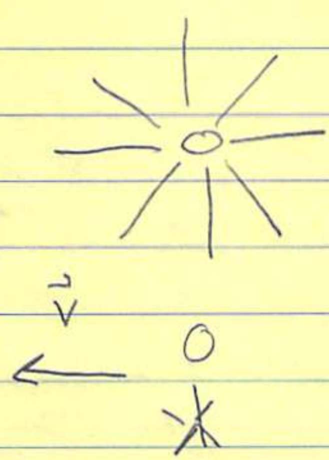
$$\underline{A}^{\mu}(\underline{x}) = (\mathcal{L})^{\mu}{}_{\nu}\, A^{\nu}(x)$$

$$= (\mathcal{L})^{\mu}{}_{\nu}\, A^{\nu}(\mathcal{L}^{-1}\underline{x})$$

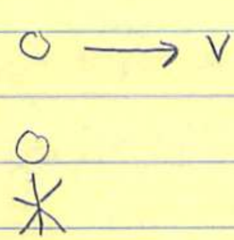
Now

$$A^{\mu} = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix} = \begin{pmatrix} \frac{q}{4\pi R} \\ \vec{0} \end{pmatrix}$$

The picture is



And we boost to person's frame



$$\begin{pmatrix} \underline{A}^0 \\ \underline{A}^x \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} q/4\pi R \\ 0 \end{pmatrix}$$

$$\underline{A}^0 = \frac{\gamma q}{4\pi R}$$

$$\underline{A}^1 = \gamma \frac{\beta q}{4\pi R}$$

Now

$$R = (x^2 + b^2)^{1/2}$$

$$\underline{X}^\mu = (\alpha)^\mu{}_\nu X^\nu \qquad X^\mu = (\alpha^{-1})^\mu{}_\nu \underline{X}^\nu$$

Or

$$\begin{pmatrix} ct \\ x \\ b \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{ct} \\ \underline{x} \\ \underline{b} \end{pmatrix}$$

$$x = -\gamma\beta ct + \gamma x = \gamma(x - vt)$$

$$b = \underline{b}$$

So

$$R = (\gamma^2 (x - vt)^2 + b^2)^{1/2}$$

Finally

$$\underline{A}^0 = \frac{\gamma q}{4\pi \left(\gamma^2 (x - vt)^2 + b^2\right)^{1/2}}$$

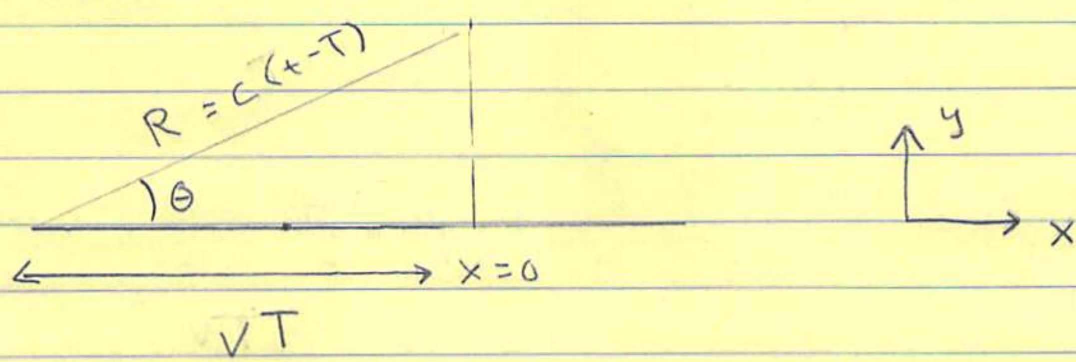
$$\underline{A}^1 = \frac{\gamma \beta q}{4\pi \left(\gamma^2 (x - vt)^2 + b^2\right)^{1/2}}$$

So

$$\underline{\underline{A^1 \Big|_{x=0} = \frac{\gamma \beta q}{4\pi \left((\gamma v t)^2 + b^2\right)^{1/2}}}}$$

$(t, r) = (t, 0, b)$

b)



$$\vec{A} = \frac{e}{4\pi} \left[\frac{\vec{v}/c}{(R - n \cdot \beta R)} \right]_{ret}$$

$$T = t - \frac{|\vec{r} - \vec{r}_*(T)|}{c}$$

Now

$$V = \beta\,\hat{x} = (\beta, 0, 0) \qquad \vec{r} = (0, b, 0)$$

$$\vec{r}_*(T) = (\beta T, 0, 0)$$

$$\sqrt{(\vec{r} - \vec{r}_*(T))^2} = (r^2 + (\beta T)^2)^{1/2}$$

So

$$T = t - \frac{(r^2 + (\beta T)^2)^{1/2}}{c}$$

Solving for T

$$T = \gamma^2 \left(t \pm \sqrt{(b/\gamma)^2 + (\beta t)^2} \right)$$

$$= \gamma^2 \left(t \pm \frac{1}{\gamma} \sqrt{b^2 + (\gamma \beta t)^2} \right)$$

We want the negative root since $T < t$ for the retarded time.

Now

$$\frac{1}{R - R\,\vec{n}\cdot\vec{\beta}} = \frac{1}{(t-T) - \vec{R}\cdot\hat{x}\beta} = \frac{1}{(t-T) + \beta^2 T}$$

We used $\vec{R} = \vec{r} - \vec{r}_*$

$$(\vec{r} - \vec{r}_*) \cdot \hat{x}\beta = -\beta^2 T$$

Evaluating $[1/R(1-\vec{n}\cdot\vec{\beta})]_{ret}$

$$\frac{1}{(t-T)+\beta^2 T} = \frac{1}{((b/\gamma)^2 + (\beta t)^2)^{1/2}} = \frac{\gamma}{(b^2 + (\gamma\beta t)^2)^{1/2}}$$

Thus $A^x = e/4\pi \; \beta / [R - \vec{R}\cdot\vec{n}\beta]_{ret}$

$$A^x = \frac{e}{4\pi} \frac{\gamma\beta}{[b^2 + (\gamma\beta t)^2]^{1/2}}$$

c)

$$\varphi = \frac{e}{4\pi}\left[\frac{1}{R(1-n\cdot\beta)}\right]_{ret}$$

$$\vec{A} = \frac{e}{4\pi}\left[\frac{\vec{\beta}}{R(1-n\cdot\beta)}\right]_{ret}$$

Use

$$u^\mu = (\gamma c, \gamma\vec{\beta})$$

$$\Delta X^\mu = X^\mu_{obs} - X^\mu_{emission}$$

$$= (c(t-T), \vec{r} - \vec{r}_*(T))$$

So

$$|\vec{R}| = c(t - T)$$

$$\vec{R} = \vec{r} - \vec{r}_* = R\vec{n}$$

So

$$R(1 - n \cdot \beta) = -\Delta X_\mu \frac{u^\mu}{\gamma_c}$$

Then

$$\varphi = \frac{e}{4\pi}\left[\frac{-\gamma c}{u \cdot \Delta X}\right]_{ret}$$

$$\vec{A} = \frac{e}{4\pi}\left[\frac{-\gamma \vec{u}}{u \cdot \Delta X}\right]_{ret}$$

Or

$$A^\mu = -\frac{e}{4\pi}\left[\frac{u^\mu}{u \cdot \Delta X}\right]_{ret}$$

Now the retarded condition means

$$T = t - \frac{R}{c} \quad \text{i.e} \quad -(t - T)^2 + \frac{R^2}{c^2} = 0$$

or $$\Delta X_\mu \Delta X^\mu = 0$$

Problem 2. Radiation during perpendicular acceleration

Consider an ultrarelativistic particle of velocity β experiencing an acceleration $a_\perp$ perpendicular to the direction of motion. Here $a_\perp$ points along the x-axis and $\boldsymbol{\beta}$ points along the z-axis.

(a) Show that the energy radiated per retarded time is approximately

$$\begin{aligned} \frac{dW}{dTd\Omega} =& \frac{e^2}{16\pi^2 c^3} \frac{a_\perp^2}{(1-\beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] && (6) \\ \simeq& \frac{e^2}{2\pi^2 c^3} \frac{a_\perp^2}{(1+(\gamma\theta)^2)^3} \left[1 - \frac{4(\gamma\theta)^2\cos^2\phi}{(1+(\gamma\theta)^2)}\right] && (7) \end{aligned}$$

In the first equality, I give the full answer without approximation, but I will only grade the second approximate result.

Hint, in working out this radiation pattern you might (as a start) show without approximation that

$$|\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}|^2 = (1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^2 a^2 - (\boldsymbol{n}\cdot\boldsymbol{a})^2(1-\beta^2) \tag{8}$$

by using the "b(ac)-(ab)c" rule. Then select a coordinate system were

$$\begin{aligned} \boldsymbol{\beta} =& (0, 0, \beta) && (9) \\ \boldsymbol{a} =& (a_\perp, 0, 0) && (10) \\ \boldsymbol{n} =& (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) && (11) \end{aligned}$$

(b) Work in a ultra-relativistic approximation, and compute the total power by integrating over the solid angle (as done in class) to show that you obtain the appropriate relativistic Larmour result

$$\frac{dW}{dT} = \text{come on \dots you know it \dots right?} \tag{12}$$

Problem 3. An oscillator radiating

(a) Determine the time averaged power radiated per unit sold angle for a *non-relativistic charge* moving along the z-axis with instantaneous position, $z(T) = H\cos(\omega_o T)$.

(b) Now consider relativistic charge executing simple harmonic motion. Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = \frac{e^2}{16\pi^2}\frac{c\beta^4}{H^2}\frac{\sin^2\theta\cos^2(\omega_o T)}{(1+\beta\cos\Theta\sin\omega_o T)^5} \tag{13}$$

Here $\beta = \omega_o H/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

(c) In the relativistic limit the power radiated is dominated by the energy radiated during a short time interval around $\omega_o T = \pi/2,\ 3\pi/2,\ 5\pi/2,\ \ldots$. Explain why. Where does the outgoing radiation point at these times.

(d) Let ΔT denote the time deviation from one of this discrete times, *e.g.* $T = 3\pi/(2\omega_o) + \Delta T$. Show that close to one of these time moments:

$$\frac{dP(\Delta T)}{d\Omega} = \frac{dW}{d\Delta T\,d\Omega} \simeq \frac{2e^2}{\pi^2}\frac{c\beta^4}{H^2}\gamma^6\frac{(\gamma\omega_o\Delta T)^2(\gamma\theta)^2}{(1+(\gamma\theta)^2+(\gamma\omega_o\Delta T)^2)^5} \tag{14}$$

(e) By integrating the results of the previous part over the ΔT of a single pulse, show that the time averaged power is

$$\overline{\frac{dP(T)}{d\Omega}} = \frac{e^2}{128\pi^2}\frac{c\beta^4}{H^2}\gamma^5\frac{5(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}} \tag{15}$$

(f) Make rough sketches of the angular distribution for non-relativistic and relativistic motion.

An Oscillator - Radiating

a) $a = -\omega_0^2 H \cos \omega_0 t$

Then using the larmour result

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 c^3} a^2 \sin^2\theta$$

So with $a(t)$

$$\overline{\frac{dP}{d\Omega}} = \frac{q^2}{16\pi^2 c^3} (\omega_0^2 H)^2 \sin^2\theta \; \overline{\cos^2 \omega_0 t}$$

$$= \frac{q^2}{16\pi^2 c^3} (\omega_0^2 H)^2 \sin^2\theta \cdot \frac{1}{2}$$

Using $\frac{\omega_0 H}{c} = \beta_0$

$$\frac{d\bar{P}}{d\Omega} = \frac{q^2}{16\pi^2} \frac{c\beta_0^4}{H^2} \sin^2\theta \cdot \frac{1}{2}$$

b) For a charged particle:

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = c\,|r\tilde{E}_{rad}|^2 \frac{dt}{dT}$$

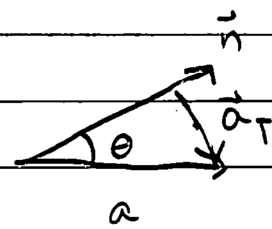
$$= \frac{q^2}{16\pi^2 c^3} \frac{|\vec{n}\times\vec{n}\times\vec{a}\,|^2}{(1-\vec{n}\cdot\vec{\beta}\,)^5} \quad ★★$$

piece of a transverse to $\vec{n}$

For $\vec{z} = H\cos\omega_0 T\,\hat{z}$ $\qquad$ $\vec{n}\times\vec{n}\times\vec{a} = -\vec{a}_T$

$\vec{a} = -\omega_0^2 H\cos\omega_0 T\,\hat{z}$ $\qquad$ $|\vec{n}\times\vec{n}\times\vec{a}\,| = |\vec{a}_T| = a\sin\theta$

$\beta = -\frac{\omega_0 H}{c}\sin\omega_0 T\,\hat{z} \equiv -\beta_0 \sin\omega_0 T\,\hat{z}$



Then

$$\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{a^2\sin^2\theta}{(1-\vec{\beta}\cdot\vec{n}\,)}$$

Using $\vec{n} = (\sin\theta, 0, \cos\theta)$, $\vec{\beta}\cdot\vec{n} = -\overbrace{\frac{\omega_0 H}{c}}^{\equiv\beta_0}\sin\omega_0 T\cos\theta$,

and $\vec{a}^2 = (\omega_0^2 H)^2\cos\omega_0 T$, so putting these into

★★ we have

$$\boxed{\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2}\left(\frac{c\beta_0^4}{H^2}\right)\frac{\cos^2\omega_0 T\sin^2\theta}{(1+\beta_0\sin\omega_0 T\cos\theta)^5}}$$

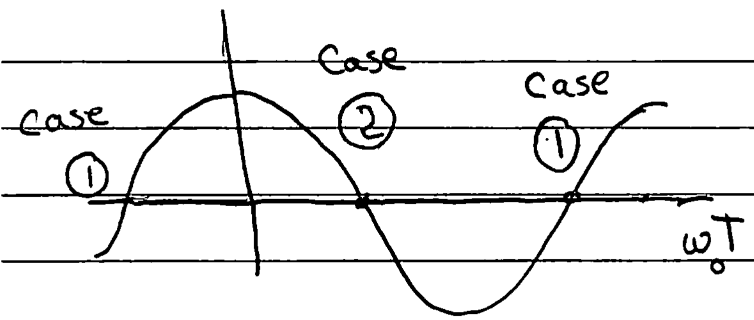
c) Looking at the denominator:

$(1 + \beta \sin\omega_0 T \cos\theta)$

① We see that this almost vanishes when $\theta \to 0$ and the time goes to $T \simeq -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$:

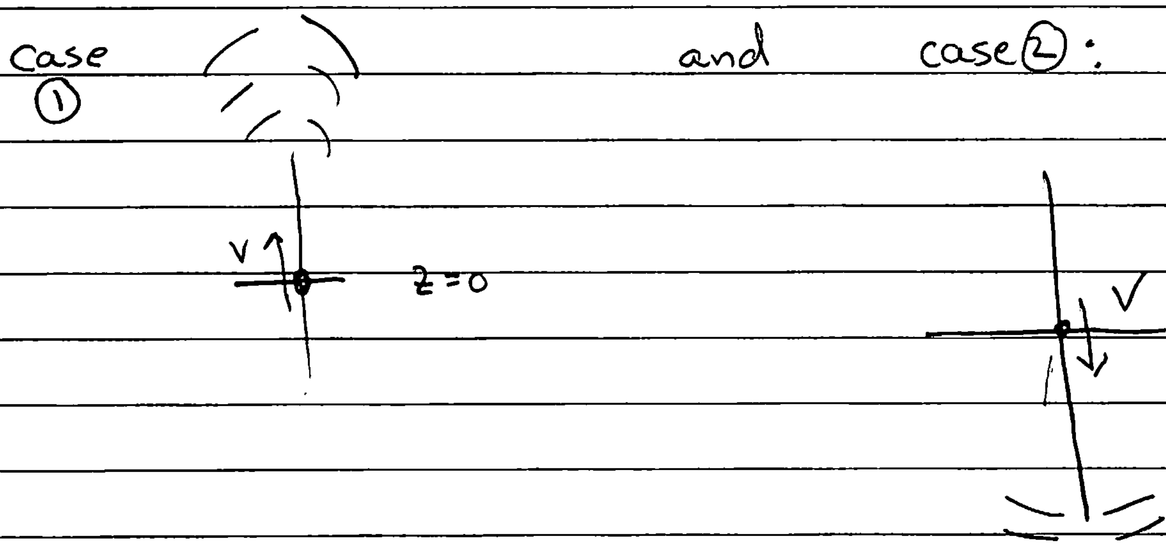
② Or, when $\theta \to \pi$ and the time goes to $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2},$ etc:

$z \equiv$ oscillator coord

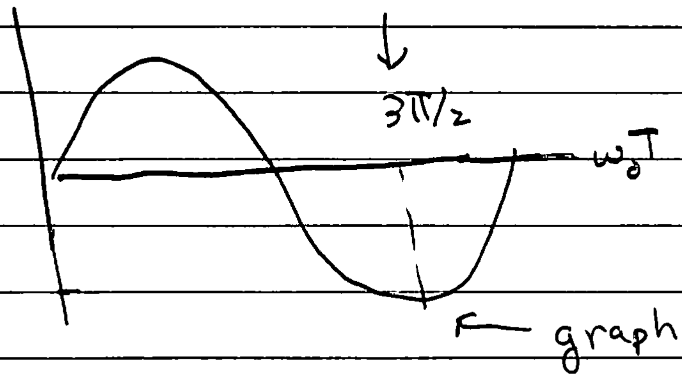


The radiation in the formula from part (b) is strongly enhaced at these points.

Thus the radiation comes predominantly from:



d) Close to one of these discrete time moments we have an approximation of the sin curve



$$\sin\omega_0 T = -1 + \frac{(\omega_0 \Delta T)^2}{2}$$

$$\cos\Theta \simeq 1 - \frac{\Theta^2}{2}$$

Then the denominator $(1+\beta\cos\Theta\cos\omega_0 T)$ is:

$$\simeq \left(1 + \beta_0\left(1-\frac{\Theta^2}{2}\right)\left(-1 + \frac{(\omega_0\Delta T)^2}{2}\right)\right)$$

$$\simeq (1-\beta_0) + \frac{\beta_0\Theta^2}{2} + \frac{\beta_0(\omega_0\Delta T)^2}{2}$$

$$\simeq \frac{1}{2\gamma^2} + \frac{\Theta^2}{2} + \frac{(\omega_0\Delta T)^2}{2}$$

since Θ^2 is small and ΔT is small we can set $\beta_0 \simeq 1$ in front of these terms

Then

$$\frac{1}{(1+\beta\cos\Theta\cos\omega_0 T)} \simeq \frac{2\gamma^2}{(1+(\gamma\Theta)^2+(\gamma\omega_0\Delta T)^2)}$$

Similarly near to $T \simeq \frac{3\pi}{2\omega_0} + \Delta T$ we approximate

$\cos\omega_0 T \simeq \omega_0 \Delta T$

and we approximate $\sin\theta \simeq \theta$.

Then

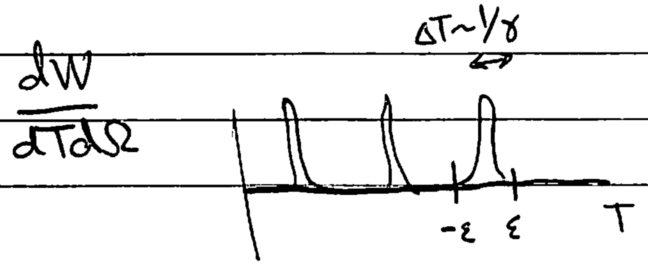
$$\frac{dW}{dT\,d\Omega} = \frac{q^2}{16\pi^2}\frac{c\beta^4}{H^2}\frac{\sin^2\theta\cos^2\omega_0 T}{(1+\beta\cos\theta\sin\omega_0 T)^5}$$

$$= \frac{q^2}{16\pi^2}\frac{c\beta^4}{H^2}\frac{(2\gamma^2)^5\,\theta^2(\omega_0\Delta T)^2}{(1+(\gamma\theta)^2+(\gamma\omega_0\Delta T)^2)}$$

$$\boxed{\frac{dW}{d\Delta T\,d\Omega} = \frac{q^2}{16\pi^2}\frac{c\beta^4}{H^2}\cdot 32\,\gamma^6\cdot\frac{(\gamma\theta)^2(\gamma\omega_0\Delta T)^2}{(1+(\gamma\theta)^2+(\gamma\omega_0\Delta T)^2)^5}}$$

e) To find the total energy in a pulse we integrate over ΔT.

$$\int_{-\varepsilon}^{\varepsilon} d\Delta T\,\frac{dW}{d\Delta T\,d\Omega} = \frac{dW_{\text{pulse}}}{d\Omega}$$



Here ε is a cutoff $\varepsilon \ll \frac{2\pi}{\omega_0}$ but much much greater than $1/\gamma$:

$$\frac{2\pi}{\omega_0}\frac{1}{\gamma} \ll \varepsilon \ll \frac{2\pi}{\omega_0}$$

$$\text{const} \equiv \frac{2q^2}{\pi^2}\frac{c\beta^4}{H^2}\gamma^6$$

So

$$\frac{dW_{pulse}}{d\Omega} = \int_{-\gamma\omega_0\varepsilon}^{\gamma\omega_0\varepsilon} \frac{d(\gamma\omega_0\Delta T)}{\gamma\omega_0}(\text{const}) \frac{(\gamma\theta)^2(\gamma\omega_0\Delta T)^2}{(1+(\gamma\theta)^2+(\gamma\omega_0\Delta T)^2)^5}$$

$$= \frac{\text{const}}{\gamma\omega_0}\int_{-\infty}^{\infty} dy \frac{(\gamma\theta)^2 y^2}{(1+(\gamma\theta)^2+y^2)^5}$$

This integral can be handled by elementary means or contour integration. Switching vars to $u \equiv \frac{y}{\sqrt{1+(\gamma\theta)^2}}$ we have:

$$\frac{dW_{pulse}}{d\Omega} = \frac{(\text{const})(\gamma\theta)^2}{\gamma\omega_0}\frac{1}{(1+(\gamma\theta)^2)^{7/2}}\underbrace{\int_{-\infty}^{\infty} du\frac{u^2}{(1+u^2)^5}}_{5\pi/128}$$

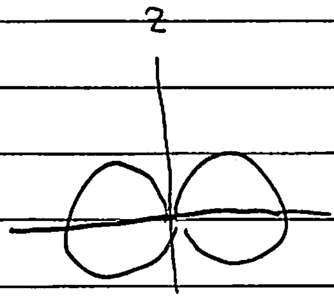
So

$$\frac{dW_{puls}}{d\Omega} = (\text{const})\frac{1}{\gamma\omega_0}\frac{5\pi}{128}\cdot\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}}$$

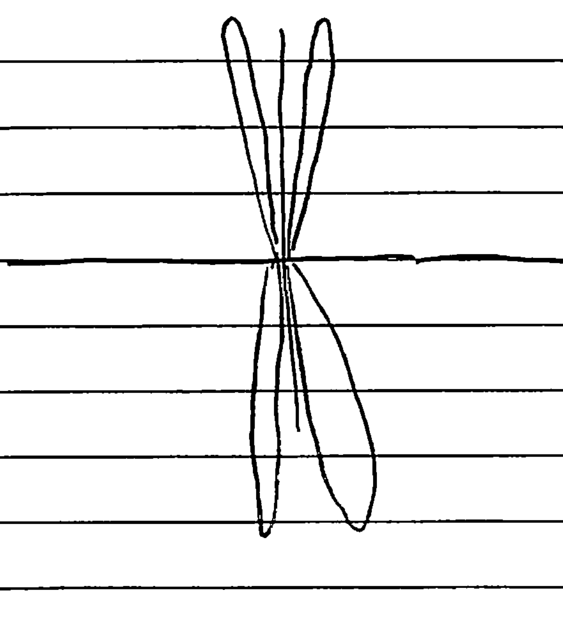
We don't want the energy in a pulse
we want time averaged power.
Multiplying $\frac{dW_{pulse}}{d\Omega}$ by the # of pulses per
sec $\frac{\omega_0}{2\pi}$, we have (restoring const)

$$\overline{\frac{dP}{d\Omega}} = \frac{q^2}{128\pi} \frac{c\beta^4}{H^2} \gamma^5 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}}$$

f) Then for the non-relativistic case:



While for the relativistic case:

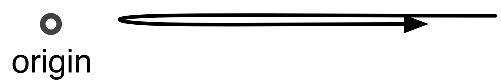


Problem 4. Radiation during a collision

A classical non-relativistic charged particle of charge q and mass m is incident upon a repulsive mechanical potential $U(r)$

$$U(r) = \frac{\mathcal{A}}{r^2},$$

so that the force on the particle is $\boldsymbol{F} = -\nabla U(r)$. The particle moves along the x-axis and strikes the central potential head on as shown below. The incident kinetic energy (i.e. the kinetic energy of the particle far from the origin) is K.

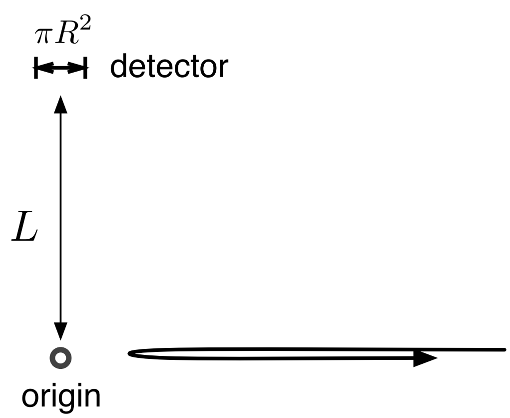


(a) (2 points) Determine the particle's classical trajectory $x(t)$. Adjust the integration constants so that the particle reaches its distance of closest approach at $t = 0$. Check that for late times $x(t)$ approaches $v_o\, t$ with the phyically correct value of v_o. Check that for small times $x(t)$ behaves as $x(t) \simeq x_o + \frac{1}{2}a_o t^2$ with the physically correct value of x_o.

(b) (4 points) Use dimensional reasoning and the Larmour formula to estimate the total energy lost to electromagnetic radiation during the collision. How does the energy lost scale with the incident velocity?

(c) (2 points) Calculate the energy lost to radiation during the collision processes. Some relevant integrals are given at the end of this problem.

Now consider a detector placed along the y-axis far from the origin as shown below. The front face of the detector has an area of πR^2, and the detector is placed at a distance L from the origin with $L \gg R$.



(d) (2 points) What is the direction of polarization of the observed light in the detector? Explain.

(e) (2 points) What is the typical frequency of the photons that are emitted at 90^o? Explain.

(f) (5 points) For the detector described above, determine the average number of photons received by the detector per unit frequency:

$$\frac{dN}{d\omega}\,. \tag{16}$$

Some relevant integrals are given at the end of the problem.

(g) (3 points) We have determined the photon radiation spectrum using classical electrodynamics. For what values of the parameters $\mathcal{A}$ and K is this approximation justified?

Useful integrals and formulas:

(a) For positive integer n, we note the integrals

$$\int_{-\infty}^{\infty} du \frac{1}{(1+u^2)^n} = \pi\, c_n \tag{17}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \ldots \tag{18}$$

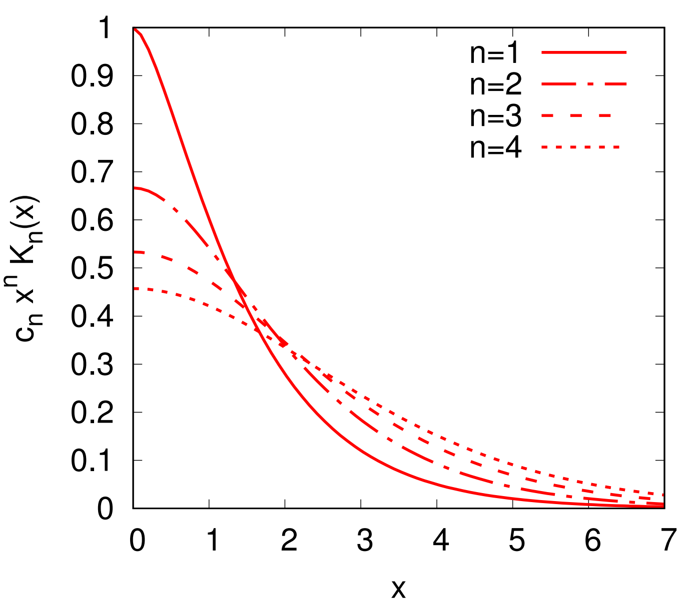
(b) For positive integers n, we note the integrals

$$\int_0^{\infty} du \frac{\cos(xu)}{(u^2+1)^{n+\frac{1}{2}}} = c_n\, x^n K_n(x) \tag{19}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{3}, \frac{1}{15}, \frac{1}{105}, \ldots \tag{20}$$

and $K_n(x)$ are the modified Bessel functions, and the RHS of Eq. (19) is illustrated below



Solution

(a) It is convenient to use dimensionless variables. The dimensional constants for the classical problem are

$$K, \mathcal{A}, m\,, \tag{6}$$

from which we can select a unit for meters, seconds, and kilograms. The unit of velocity can be taken as

$$K = \frac{1}{2} m v_o^2 \Longrightarrow v_o \equiv \sqrt{\frac{2K}{m}}\,, \tag{7}$$

which phyiscally is the velocity as $r \to \infty$. The unit of meters is

$$K = \frac{\mathcal{A}}{x_o^2} \Longrightarrow x_o \equiv \sqrt{\frac{\mathcal{A}}{K}}\,, \tag{8}$$

which (by energy conservation) is the distance of closest approach[2]. The unit of seconds is therefore

$$t_o \equiv \frac{x_o}{v_o} \equiv \frac{\sqrt{(m\mathcal{A})/2}}{K}\,. \tag{9}$$

We need to solve for the trajectory $x(t)$. The velocity is given by the first integral (i.e. enegy conservation)

$$\frac{1}{2} m v^2(t) + \frac{\mathcal{A}}{x(t)^2} = K\,. \tag{10}$$

Switching to dimensionless variables,

$$\bar{v} = \frac{v}{v_o}\,, \qquad \bar{x} = \frac{x(t)}{x_o}\,, \tag{11}$$

the dimensionless form of energy conservation reads

$$\bar{v}^2 + \frac{1}{\bar{x}^2} = 1\,. \tag{12}$$

Solving Eq. (12) for $\bar{v}$ we have

$$\bar{v} = \sqrt{1 - \frac{1}{\bar{x}^2}}\,. \tag{13}$$

Finally, we write $\bar{v} = d\bar{x}/d\bar{t}$ and integrate Eq. (13) to find

$$\sqrt{\bar{x}^2 - 1} = \bar{t} + \text{constant}\,. \tag{14}$$

[2] Note that this is the distance of closest approach in the absence of energy loss due to radiation. In the limit of classical electrodynamics one first determines the trajectories of charged particles (ignoring the radiation), and then soves for the subsequent radiation. This is in effect ignoring radiations back reaction.

We choose the integration constant to be zero, so that at $t = 0$ the trajectory is at the turning point $\bar{x} = 1$ and find

$$\sqrt{\bar{x}^2 - 1} = \bar{t} \qquad \text{or} \qquad \bar{x}(\bar{t}) = \sqrt{1 + \bar{t}^2} \tag{15}$$

Restoring units, the trajectory is

$$x(t) = \sqrt{\frac{2Kt^2}{m} + \frac{\mathcal{A}}{K}}\,. \tag{16}$$

It is easy to check that this trajectory satisfies the appropriate limits.

(b) The energy lost to radiation is

$$E_{\rm loss} = \int_{-\infty}^{\infty} dt\, \frac{q^2}{4\pi} \frac{2a^2}{3c^3} \tag{17}$$

We need to use dimensional reasoning to estimate a and the time interval over which the acceleration is active.

Using the dimensional analysis of the previous section, the integral is of order

$$\int dt\, a^2 \sim \frac{v_o^2}{t_o}\,. \tag{18}$$

and thus

$$E_{\rm loss} \sim \frac{q^2}{4\pi t_o} \frac{v_o^2}{c^3} \sim \frac{q^2 K^2}{m\sqrt{Am}c^3}\,. \tag{19}$$

The energy lost scales as the velocity to the fourth power, $K^2 \propto v_o^4$.

(c) We next evaluate the integral in Eq. (17) precisely. For reference we record the acceleration:

$$a(t) = \frac{v_o}{t_o} \frac{d^2\bar{x}}{d\bar{t}^2} = \frac{v_o}{t_o} \frac{1}{(1 + \bar{t}^2)^{3/2}}\,. \tag{20}$$

The relevant integral is

$$\int_{-\infty}^{\infty} dt\, a^2 = \frac{v_o^2}{t_o} \int_{-\infty}^{\infty} d\bar{t} \left(\frac{d^2\bar{x}}{d\bar{t}^2}\right)^2 = \frac{v_o^2}{t_o} \int_{-\infty}^{\infty} \frac{d\bar{t}}{(1+\bar{t}^2)^3} = \frac{v_o^2}{t_o} \left(\frac{3\pi}{8}\right)\,. \tag{21}$$

The energy lost is therefore

$$E_{\rm loss} = \frac{q^2}{4\pi} \frac{2}{3c^3} \int_{-\infty}^{\infty} dt\, a^2\,, \tag{22}$$

$$= \frac{q^2}{4\pi t_o} \frac{v_o^2}{c^3} \left(\frac{\pi}{4}\right)\,. \tag{23}$$

(d) The radiation electric field is

$$\boldsymbol{E}_{\rm rad}(t,\boldsymbol{r}) = \frac{q}{4\pi r c^2}\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{a}(t_e)\,, \tag{24}$$

where the emission time is

$$t_e = t - \frac{r}{c}\,. \tag{25}$$

For the problem at hand $\boldsymbol{a} = a(t)\hat{\boldsymbol{x}}$ and thus

$$\boldsymbol{n}\times\boldsymbol{n}\times\hat{\boldsymbol{x}} = -\hat{\boldsymbol{x}}\,. \tag{26}$$

So the radiation field is polarized in the $-\hat{\boldsymbol{x}}$ direction.

(e) The typical frequency is given by dimension reasoning

$$\omega_o \sim \frac{1}{t_o} \tag{27}$$

(f) To determine the yield of photons, we Fourier transform the radiation field and square this Fourier transform. The Fourier transform of the electric field (in the $-\hat{\boldsymbol{x}}$ direction) reaching the detector

$$E_{\rm rad}(\omega, r) = \frac{q}{4\pi r c^2}\int_{-\infty}^{\infty} dt e^{-i\omega t} a(t_e)\,. \tag{28}$$

After switching to variables to integrate over the emission time,

$$e^{i\omega t} = e^{i\omega(t_e + r/c)} = e^{ikr}e^{i\omega t_e}, \qquad k \equiv \frac{\omega}{c}, \tag{29}$$

the integral reads

$$E_{\rm rad}(\omega, r) = \frac{q\, e^{ikr}}{4\pi r c^2}\int_{-\infty}^{\infty} dt_e e^{-i\omega t_e} a(t_e)\,. \tag{30}$$

Thus, after switching to dimensionless variables

$$\bar{\omega} = \omega t_o \qquad \bar{t}_e = \frac{t_e}{t_o}\,, \tag{31}$$

Thus we find

$$\begin{aligned} E_{\rm rad}(\omega, r) =& \frac{q\, e^{ikr}}{4\pi r c^2} v_o \int_{-\infty}^{\infty} d\bar{t}_e e^{-i\bar{\omega}\bar{t}_e}\, \frac{1}{(1+\bar{t}_e^2)^{3/2}}\,, & (32)\\ =& \frac{q\, e^{ikr}}{4\pi r c^2} v_o \left[2\bar{\omega} K_1(\bar{\omega})\right]\,. & (33) \end{aligned}$$

Squaring the radiation field, we find the yield of photons

$$\hbar\omega \frac{dN}{d\omega d\Omega} = \frac{c}{\pi}|r E_{\rm rad}(\omega, r)|^2\,. \tag{34}$$

Assembling the ingredients, and expressing the result in terms of the fine structure constant $\alpha = q^2/(4\pi\hbar c) = 1/137$, we find

$$\frac{dN}{d\omega d\Omega} = \frac{\alpha}{4\pi^2}\left(\frac{v_o}{c}\right)^2 \frac{1}{\omega}\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 . \tag{35}$$

The solid angle is $\Delta\Omega = \pi R^2/L^2$, and thus we find

$$\frac{dN}{d\omega} = \frac{\pi R^2}{L^2}\frac{\alpha}{4\pi^2}\left(\frac{v_o}{c}\right)^2 \frac{1}{\omega}\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 . \tag{36}$$

In the low frequency limit the term in brackets approaches

$$\left[2\bar{\omega}K_1(\bar{\omega})\right]^2 \to 2^2 , \tag{37}$$

and thus in low frequency limit we find

$$\frac{dN}{d\omega d\Omega} = \frac{\alpha}{4\pi^2}\left(\frac{2v_o}{c}\right)^2 \frac{1}{\omega} . \tag{38}$$

Notice that this expression is independent of $\mathcal{A}$, and is in fact indentical to the radiation for impulsive scattering where $v(t)$ changes instantaneously:

$$\boldsymbol{v}_{\rm impulse}(t) = \begin{cases} -v_o\,\hat{\boldsymbol{x}} & t < 0 \\ v_o\,\hat{\boldsymbol{x}} & t > 0 \end{cases} . \tag{39}$$

Indeed, in the low frequency limit the radiated waves do not have the temporal resolution to resolve events of order the collision time t_o. Thus, as far as the radiation of these low frequency waves is concerned, the collision happens instantaneously.

(g) To determine the validity of the classical approximation, we note that the typcal frequency is $1/t_o$. The energy of the emitted photon has to be small compared to the kinetic energy of the particle for the classical approximation to be valid

$$\hbar\omega \ll K . \tag{40}$$

With $\frac{1}{t_o} = \frac{K}{\sqrt{(m\mathcal{A})/2}}$, we find

$$\frac{2\hbar^2}{m\mathcal{A}} \ll 1 . \tag{41}$$

Problem 5. Periodic pulses

Consider a periodic motion that repeats itself with period $\mathcal{T}_o$. Show that the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental, $\omega_o = 2\pi/\mathcal{T}_o$.

Let the electric field from a single pulse (or period) be $E_1(t)$, *i.e.* where $E_1(t)$ is non-zero between 0 and $\mathcal{T}_o$ and vanishes elsewhere, $t < 0$ and $t > \mathcal{T}_o$. Let $E_1(\omega)$ be its fourier transform.

(a) Suppose that the wave form repeats once so that two pulses are received. $E_2(t)$ consists of the first pulse $E_1(t)$, plus a second pulse, $E_2(t) = E_1(t) + E_1(t - \mathcal{T}_o)$. Show that the Fourier transform and the power spectrum is

$$E_2(\omega) = E_1(\omega)\,(1 + e^{i\omega\mathcal{T}_o}) \qquad |E_2(\omega)|^2 = |E_1(\omega)|^2\,(2 + 2\cos(\omega\mathcal{T}_o)) \tag{21}$$

(b) Now suppose that we have n (with n odd) arranged almost symmetrically around $t = 0$, *i.e.*

$$E_n(t) = E_1(t+(n-1)\mathcal{T}_o/2)+\ldots+E_1(t+\mathcal{T}_o)+E_1(t)+E_1(t-\mathcal{T}_o)+\ldots E_1(t-(n-1)\mathcal{T}_o/2)\,, \tag{22}$$

so that for $n = 3$

$$E_3(t) = E_1(t + \mathcal{T}_o) + E_1(t) + E_1(t - \mathcal{T}_o)\,. \tag{23}$$

Show that

$$E_n(\omega) = E_1(\omega)\,\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)} \tag{24}$$

and

$$|E_n(\omega)|^2 = |E_1(\omega)|^2\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right)^2 \tag{25}$$

(c) By taking limits of your expressions in the previous part show that after n pulses, with $n \to \infty$, we find

$$E_n(\omega) = \sum_m E_1(\omega_m)\frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m) \tag{26}$$

and

$$|E_n(\omega)|^2 = \underbrace{n\mathcal{T}_o}_{\text{total time}} \times \sum_m |E_1(\omega_m)|^2\frac{2\pi}{\mathcal{T}_o^2}\delta(\omega - \omega_m) \tag{27}$$

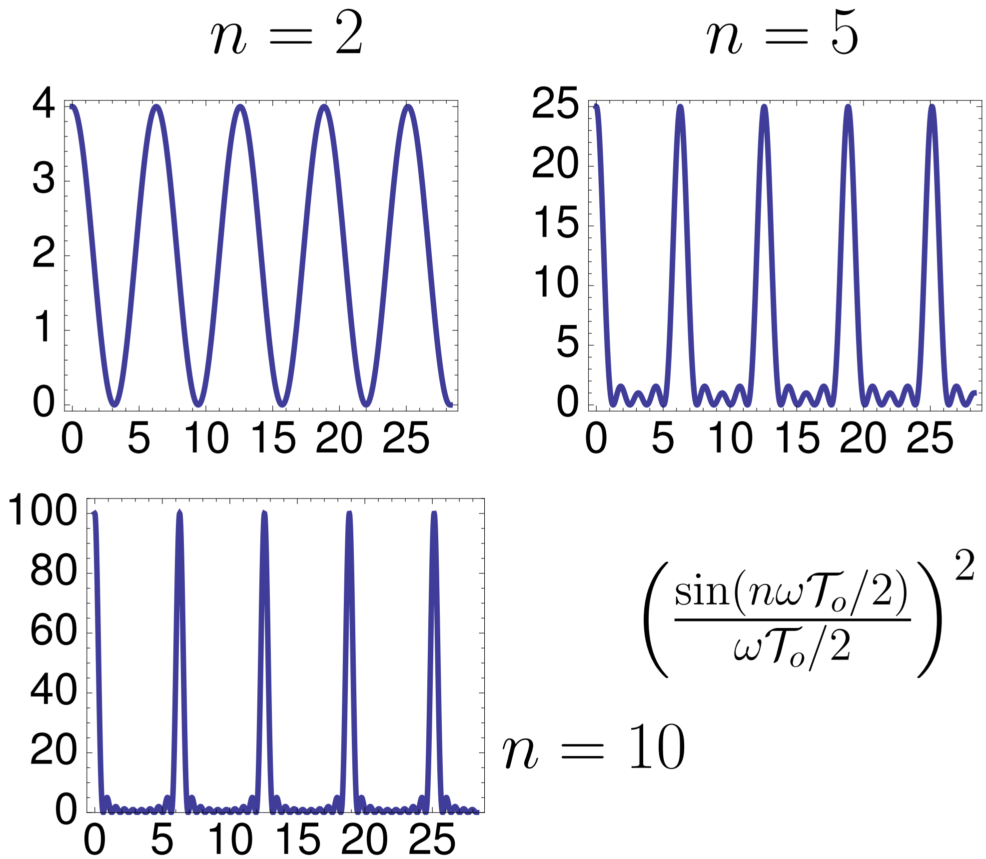
where $\omega_m = 2\pi m/\mathcal{T}_o$.

Remark We have in effect shown that if we define

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o)\,. \tag{28}$$

Then the Fourier transform of $\Delta(t)$ is

$$\hat{\Delta}(\omega) = \sum_n e^{-i\omega n\mathcal{T}_o} = \sum_m \frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m)\,. \tag{29}$$


$n = 2$
$n = 5$
$n = 10$
$\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\omega\mathcal{T}_o/2}\right)^2$

(d) Show that a general expression for the time averaged power radiated per unit solid angle into each multipole $\omega_m \equiv m\omega_o$ is:

$$\frac{dP_m}{d\Omega} = \frac{|rE(\omega_m)|^2}{\mathcal{T}_o^2} \tag{30}$$

Or

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2\omega_o^4 m^2}{32\pi^4 c^3} \left| \int_0^{\mathcal{T}_o} \mathbf{v}(T) \times \boldsymbol{n} \exp\left[i\omega_m (T - \frac{\boldsymbol{n} \cdot r_*(T)}{c}) \right] \right|^2 \mathrm{d}T \,, \tag{31}$$

Here $d\hat{P}_m/d\Omega$ is defined so that over along time period $\Delta\mathcal{T}$, the energy per solid angle is

$$\frac{dW}{d\Omega} = \Delta\mathcal{T} \sum_{m=1}^{\infty} \frac{d\hat{P}_m}{d\Omega} \tag{32}$$

Also note that we are summing only over the positive values of m which is different from how we had it in class:

$$\frac{d\hat{P}_m}{d\Omega} \equiv \frac{dP_m}{d\Omega} + \frac{dP_{-m}}{d\Omega} \tag{33}$$

Periodic Pulses

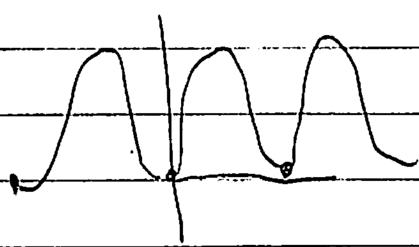
a) $$E(\omega) = \int dt\, e^{i\omega t} E(t)$$

$$= \int dt\, e^{i\omega t}\, E_1(t) + E_1(t - T_0)$$

$$= E_1(\omega) + \int d\Delta t\, e^{i\omega \Delta t + T_0} E_1(\Delta t)$$

$$= E_1(\omega) + e^{i\omega T_0} E_1(\omega)$$

b) For a set of pulses



$$E_N(\omega) = \int dt\, e^{i\omega t} \Big[E_1(t + \tfrac{(n-1)}{2} T_0) + \dots\ E_1(t) + \dots$$

$$E_1(t - \tfrac{(n-1)}{2} T_0) \Big]$$

$$= E_1(\omega) \big(e^{-i\omega(n-1)T_0/2} + \dots + 1 + \dots\ e^{i\omega(n-1)T_0/2} \big]$$

For three pulses

$$E_3(\omega) = E_1(\omega) \left[e^{-i\omega T_0} + 1 + e^{i\omega T_0} \right]$$

Pulling out a common factor

$$E_n(\omega) = E_1(\omega)\, e^{-i\omega(n-1)T_0/2} \left[1 + \dots\dots e^{i\omega(n-1)T_0}\right]$$

Using

$$(1-x)(1+x+\dots x^{n-1}) = (1-x^n)$$

$$E_n(\omega) = E_1(\omega)\, e^{-i\omega(n-1)T_0/2} \left(\frac{1 - e^{+i\omega n T_0}}{1-e^{i\omega T_0}}\right)$$

$$= E_1(\omega) \left(\frac{e^{-i\omega(n-1)T_0/2} - e^{+i\omega n T_0/2}}{e^{-i\omega T_0/2} - e^{+i\omega T_0/2}}\right)$$

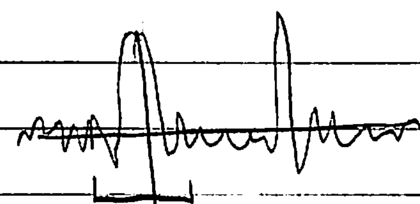
$$E_n(\omega) = E_1(\omega)\, \frac{\sin(\omega n T_0/2)}{\sin(\omega T_0/2)}$$

Taking limits

$$E_N(\omega) \xrightarrow[N \to \infty]{} E_1(\omega) \; \frac{2\pi}{T_0} \sum_m \delta(\omega - \omega_m)$$

To see this we firsit recognize that the function

$$\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)}$$



$-\varepsilon < \Delta\omega < \varepsilon$

is 2π periodic. Then near the peaks we expand the denominator but not the numerator, since although $\Delta\omega$ is small, $n\Delta\omega$ is not small. Near the peak we have

$$E_N(\omega) \simeq E_1(\omega) \; \frac{\sin(n\Delta\omega T_0/2)}{(\Delta\omega T_0/2)}$$

And calculate the integral

$$I = \int_{-\varepsilon}^{\varepsilon} d\Delta\omega \left[\frac{\sin(n\Delta\omega T_0/2)}{\Delta\omega T_0/2} \right]$$

$$X \equiv n\frac{\Delta\omega T_0}{2}$$

$$I = \frac{2}{T_0} \int_{-n\varepsilon T_0/2}^{+n\varepsilon T_0/2} dx \left(\frac{\sin x}{x} \right)$$

$$I = 2\pi / T_0$$

We have thus established that

$$\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)}$$

becomes infinitely narrow but has a finite Integral, i.e. it is a delta-fcn representation

$$\frac{\sin(n\omega T_0/2)}{\sin(\omega T_0/2)} \xrightarrow[n\to\infty]{} \frac{2\pi}{T_0}\,\delta(\omega-\omega_m)$$

for all values of $\omega_m = 2\pi m/T_0$

So

$$\boxed{E_N(\omega) \to E_1(\omega)\sum_m \frac{2\pi}{T_0}\,\delta(\omega-\omega_m)}$$

Then similarly

$$|E_N(\omega)|^2 = |E_1(\omega)|^2 \left(\frac{\sin n\omega T_0/2}{\sin \omega T_0/2}\right)^2$$

Plotting this squared fcn

Periodic Pulses

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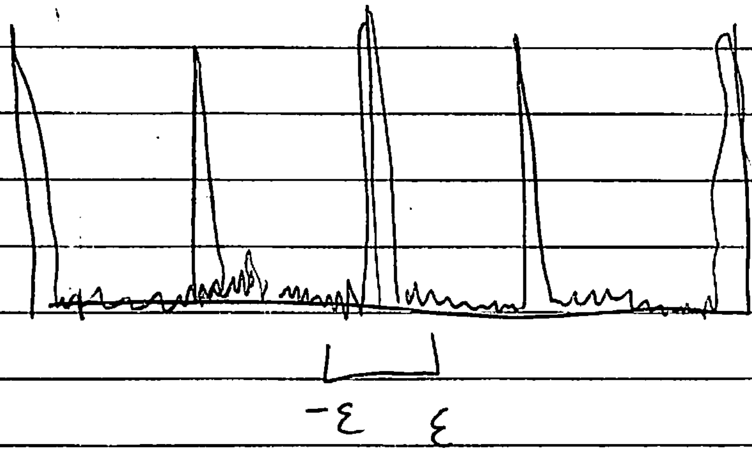


We integrate over the spikes

$$I = \int_{-\varepsilon}^{\varepsilon} d\omega \left(\frac{\sin n\omega T_0/2}{\sin \omega T_0/2} \right)^2 = \int_{-\varepsilon}^{\varepsilon} d\omega \left(\frac{\sin n\omega T_0/2}{\omega T_0/2} \right)^2$$

where we expand

$$\sin \omega T_0/2 \simeq \omega T_0/2$$

We now switch vars to $x = \omega n T_0/2$ taking $n \to \infty$ with ε fixed with $x \equiv \omega n T_0/2$

$$I = n \frac{2}{T_0} \int_{-\infty}^{\infty} dx \left(\frac{\sin x/2}{x/2} \right)^2$$

$$I = n \frac{2\pi}{T_0}$$

So

$$|E_N(\omega)|^2 = n \, |E_1(\omega)|^2 \frac{2\pi}{T_0} \sum_m \delta(\omega - \omega_m)$$

Thus

$$\boxed{\frac{|E_N(\omega)|^2}{(\text{Total time})} = |E_1(\omega)|^2 \frac{2\pi}{T_0^2} \sum_m \delta(\omega - \omega_m)}$$

d) Now

$$(2\pi) \frac{dW}{d\omega \, d\Omega} = c \, |E_N(\omega)|^2 \cdot r^2$$

Then

$$\frac{dP_m}{d\Omega} = \int_{\omega_m - \text{bit}}^{\omega_m + \text{bit}} \frac{d\omega}{2\pi} \frac{2\pi \, dW}{d\omega \, d\Omega} \frac{1}{(\text{Total Time})}$$

$$= \int_{\omega_m - \text{bit}}^{\omega_m + \text{bit}} c \, | r E_1(\omega)|^2 \frac{2\pi}{T_0^2} \sum_m \delta(\omega - \omega_m) \frac{d\omega}{2\pi}$$

Thus

$$\frac{dP_m}{d\Omega} = c\frac{|E_1(\omega_m)|^2}{T_0^2} \cdot r^2$$

$$\star\star \quad \frac{d\hat{P}_m}{d\Omega} = \frac{dP_m}{d\Omega} + \frac{dP_{-m}}{d\Omega} = \frac{2c\,|E_1(\omega_m)|^2 r^2}{T_0^2} \quad \text{with } \omega_m > 0$$

So with:

$$\vec{E}_1(\omega_m) = \frac{q}{4\pi r c^2}(-i\omega_m)\int_0^{T_0} dT\, e^{i\omega_m(T - \frac{\vec{n}}{c}\cdot\vec{r}_*(T))}\, \vec{n}\times\vec{n}\times\vec{v}(T)$$

And note that

$$|E_1(\omega_m)|^2 = |B_1(\omega_m)|^2 \qquad (\vec{E} \text{ and } \vec{B} \text{ have equal magnitude})$$

So

$$\vec{n}\times\vec{E} = \vec{B}_1(\omega_m) = \frac{q}{4\pi r c^2}(-i\omega_m)\int_0^{T_0} dT\, e^{i\omega_m(T - \frac{\vec{n}}{c}\cdot\vec{r}_*(T))}\,(-\vec{n}\times\vec{v}(T))$$

So substituting into $\star\star$

$$\frac{d\hat{P}_m}{d\Omega} = \frac{2c|B_1(\omega_m)|^2 r^2}{T_0^2} = \frac{q^2}{16\pi^2 c^3}\frac{2\omega_m^2}{T_0^2}\left|\int_0^{T_0} dT\, e^{i\omega_m(T - \frac{\vec{n}}{c}\cdot\vec{r}_*(T))}\,\vec{n}\times\vec{v}\right|^2$$

So with $\omega_m = m\omega_0$ $\quad T_0 = \frac{2\pi}{\omega_0}$

the overall factor is

$$\frac{q^2 \omega_0^4 m^2}{32\pi^4 c^3}$$

and

$$\boxed{\frac{d\hat{P}_m}{d\Omega} = \frac{q^2 \omega_0^4 m^2}{32\pi^4 c^3} \left| \int_0^{T_0} dT \ \vec{n} \times \vec{V} \ e^{i\omega_m (T - \frac{\vec{n}}{c} \cdot r_*(T))} \right|^2}$$

Problem 6. Radiation spectrum of a SHO

(a) Show that for the simple harmonic motion of a charge discussed in Problem: An Oscillator Radiating, the average power radiated per unit solid angle in the m-th harmonic is

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2 c\beta^2}{8\pi^2 H^2} m^2 \tan^2\theta \left[J_m(m\beta\cos\theta)\right]^2 \tag{34}$$

(b) Show that in the non-relativistic limit the total power radiated is all in the fundamental and has the value

$$P = \frac{e^2}{4\pi}\frac{2}{3}\omega_o^4 \overline{H^2} \tag{35}$$

where $\overline{H^2}$ is the mean squared amplitude of the oscillation.

Problem 4 - Radiation Spectrum from SHO

a) Then we need to compute (from Prob 3)

$$\left| \int_0^{T_0} dT \, e^{i\omega_m (T - \frac{\vec{n}}{c}\cdot r_*(T))} \, n \times v(T) \right|^2$$

Using $\vec{n} = (\sin\theta, 0, \cos\theta)$:

$\beta \equiv \frac{\omega_0 H}{c}$ is the maximum velocity of the oscillator

$$\vec{r}_* = \hat{z} \, H \cos\omega_0 T$$

$$\vec{V} = -\hat{z} \, H\omega_0 \sin\omega_0 T \equiv -\hat{z} \, c\beta \sin\omega_0 T$$

And with $\vec{n} \times \vec{V} = (\sin\theta \, \hat{x} + \cos\theta \, \hat{z}) \times \hat{z} \, (-H\omega_0 \sin\omega_0 t)$

$$= +\sin\theta \, \hat{y} \, H\omega_0 \, \mathrm{sim}\, \omega T$$

So using also:

$T - \frac{\vec{n}}{c}\cdot\vec{r}_*(T) = T - \frac{H}{c}\cos\omega_0 T \cos\theta$, we need to eval:

$$I \equiv \int_0^{T_0} dT \, e^{i\omega_m (T - \frac{H}{c}\cos\omega_0 T \cos\theta)} \, \beta \sin\omega_0 T \sin\theta \, \hat{y}$$

Change vars $\alpha \equiv \omega_0 T$:

Radiation Spectrum from SHO pg. 2

So

$$\underline{I} = \frac{c\beta \sin\theta}{\omega_0} \hat{y} \int_0^{2\pi} d\alpha\, e^{im\alpha} e^{-im\beta\cos\alpha\cos\theta} \sin\alpha$$

Now integrate by parts:

$$\underline{I} = \frac{c\beta \sin\theta}{\omega_0} \hat{y} \int_0^{2\pi} d\alpha\, e^{im\alpha} \left[\frac{d}{d\alpha} e^{-im\beta\cos\alpha\cos\theta}\right] \frac{1}{im\beta\cos\theta}$$

$$\underline{I} = \frac{c}{\omega_0} \frac{\tan\theta}{im} \hat{y} \int_0^{2\pi} d\alpha\, e^{im\alpha} e^{-im\beta\cos\alpha\cos\theta} (-im)$$

$$\underline{I} = -\frac{c}{\omega_0} \tan\theta\, \hat{y} \int_0^{2\pi} d\alpha\, e^{im\alpha} e^{-im\beta\cos\alpha\cos\theta}$$

$$= -\frac{c}{\omega_0} \tan\theta\, \hat{y} \int_0^{2\pi} d\alpha\, e^{im\alpha} e^{+iu\cos\alpha} \qquad u \equiv -m\beta\cos\theta$$

Using the Fourier-Bessel expansion:

$$e^{iu\cos\alpha} = \sum_{m=-\infty}^{\infty} i^m J_m(u)\, e^{im\alpha}$$

Then

$$\underline{I} = -\frac{c}{\omega_0} \tan\theta\, \hat{y}\; i^{-m} J_{-m}(u)\, 2\pi$$

using for m integer: $J_{-m}(-x) = J_m(x)$

Find

$$I = -\frac{c}{\omega_0} \tan\theta \, \hat{y} \, i^{m} J_m(u) \cdot 2\pi$$

So

$$|I|^2 = \frac{c^2}{\omega_0^2} \tan^2\theta \left(J_m(m\beta\cos\theta) \right)^2 (2\pi)^2$$

Then using the results of problem 3

$$\frac{d\hat{P}_m}{d\Omega} = \frac{q^2}{32\pi^4} \frac{\omega_0^4 m^2}{c^3} \left[\frac{c^2}{\omega_0^2} \tan^2\theta \, J_m^2(m\beta\cos\theta) \right] (2\pi)^2$$

$$\frac{d\hat{P}}{d\Omega} = \frac{q^2}{8\pi^2} \frac{\omega_0^2 m^2}{c} \tan^2\theta \, J_m^2(m\beta\cos\theta)$$

with $\omega_0 = \frac{c\beta}{H}$ find

$$\boxed{\frac{d\hat{P}}{d\Omega} = \frac{q^2}{8\pi^2} \frac{c\beta^2 m^2}{H^2} \tan^2\theta \, J_m^2(m\beta\cos\theta)}$$

Radiation Spectrum from SHO pg. 4

b) For $\beta \ll 1$ we can use a series expansion for $J_m(m\beta\cos\theta)$

$$J_m(z) \simeq \left(\frac{z}{2}\right)^m \frac{1}{m!}$$

So the only term which contributes to order β^2 is $m=1$:

Then

$$\frac{d\hat{P}}{d\Omega} \simeq \frac{q^2}{8\pi^2} \frac{c\beta^2}{H^2} \overbrace{m^2}^{=1} \tan^2\theta \underbrace{\left(\frac{u}{2}\right)^2}_{(J_1(u))^2}$$

with $u = m\beta\cos\theta$ we have

$$\frac{dP_1}{d\Omega} = \frac{q^2}{8\pi^2} \frac{c\beta^2}{H^2} \sin^2\theta \left(\frac{\beta}{2}\right)^2$$

with $\beta = \omega_0 H / c$

maximum

$$\frac{H^2}{2} = \overline{H^2}$$

mean squared amp

$$\frac{d\hat{P}}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \omega_0^4 \sin^2\theta \left(\frac{H^2}{2}\right)$$

$$\frac{d\hat{P}_1}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \omega_0^4 \sin^2\theta \, \overline{H^2}$$

This is the Larmour result

Radiation Spectrum pg. 5

Integrating over angle

$$\boxed{P = \frac{q^2}{4\pi c^3} \frac{2}{3} \omega_0^4 \overline{H^2}}$$

Problem 7. (Optional) Energy during a burst of deceleration

A particle of charge e moves at constant velocity, βc, for $t < 0$. During the short time interval, $0 < t < \Delta t$ its velocity remains in the same direction but its speed decreases linearly in time to zero. For $t > \Delta t$, the particle remains at rest.

(a) Show that the radiant energy emitted per unit solid angle is

$$\frac{dW}{d\Omega} = \frac{e^2\beta^2}{64\pi^2 c\Delta t} \frac{(2-\beta\cos\theta)\left[1+(1-\beta\cos\theta)^2\right]\sin^2\theta}{(1-\beta\cos\theta)^4} \tag{36}$$

(b) In the limit $\gamma \gg 1$, show that the angular distribution can be expressed as

$$\frac{dW}{d\xi} \simeq \frac{e^2\beta^2}{4\pi\, c} \frac{\gamma^4}{\Delta t} \frac{\xi}{(1+\xi)^4} \tag{37}$$

where $\xi = (\gamma\theta)^2$.

(c) Show for $\gamma \gg 1$ that the total energy radiated is in agreement with the relativistic generalization of the Larmour formula.

Problem-A burst of deceleration

First we record β:

$$\beta(T) = \begin{cases} \beta_0 \\ \beta_0\left(1 - \dfrac{T}{\Delta t}\right) \\ 0 \end{cases} \Rightarrow \quad \frac{a}{c} = \frac{\beta_0}{\Delta T}$$

$$\gamma_0 \equiv \frac{1}{\sqrt{1-\beta_0^2}}$$

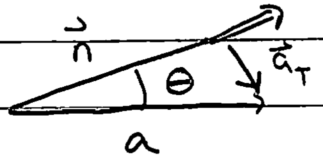
Then:

$$\frac{dW}{dT\,d\Omega} = c\,|rE_{rad}|^2 \frac{dt}{dT}$$

Where:

$$rE_{rad} = \frac{q}{4\pi c^2} \frac{n \times (n-\beta) \times a}{(1 - n\cdot\beta)^3}$$

So since $\vec{a}$ is parallel to $\vec{\beta}$, $\vec{\beta} \times \vec{a} = 0$, and



$$n \times n \times \vec{a} = -\vec{a}_T \Rightarrow |\vec{a}_T| = a\sin\theta$$

$$\frac{dW}{dT\,d\Omega} = \frac{q^2}{16\pi^2 c^3}\left(\frac{c\beta_0}{\Delta T}\right)^2 \frac{\sin^2\theta}{(1 - \beta(T)\cos\theta)^5}$$

$$= \;\text{—}\,/\!/\,\text{—}\; \frac{\sin^2\theta}{\left(1 - \beta_0\cos\theta + \dfrac{\beta_0 T}{\Delta T}\cos\theta\right)^5}$$

To find, we integrate

$$\frac{dW}{d\Omega} = \int_0^{\Delta T} \frac{dW}{dT\,d\Omega}\,dT$$

Using

$$\int \frac{dT}{\left(1-\beta_0\cos\theta + \frac{\beta_0\cos\theta}{\Delta T}\,T\right)^5} = \frac{-1}{4}\,\frac{1}{\left(1-\beta_0\cos\theta + \frac{\beta_0 T\cos\theta}{\Delta T}\right)^4} \times \frac{\Delta T}{\beta_0\cos\theta}\Bigg|_0^{\Delta T}$$

$$= \frac{1}{4}\,\frac{\Delta T}{\beta_0\cos\theta}\left[\frac{1}{(1-\beta_0\cos\theta)^4} - 1\right]$$

So collecting all factors:

$$\boxed{\frac{dW}{d\Omega} = \frac{q^2}{64\pi^2 c}\,\frac{\beta^2}{\Delta T}\left[\frac{1}{\beta\cos\theta}\left(\frac{1}{(1-\beta\cos\theta)^4} - 1\right)\right]\sin^2\theta}$$

"Simplifying" the term in square brackets using:

$$1-(1-\beta\cos\theta)^4 = \left[4\beta\cos\theta - 6(\beta\cos\theta)^2 + 4(\beta\cos\theta)^3 - (\beta\cos\theta)^4\right]$$

$$= \beta\cos\theta\,(2-\beta\cos\theta)\left(1+(1-\beta\cos\theta)^2\right)$$

We find

$$\frac{dW}{d\Omega} = \frac{q^2}{64\pi^2 c}\frac{\beta_0^2}{\Delta T}\left[\frac{(2-\beta_0\cos\theta)(1+(1-\beta_0\cos\theta)^2)}{(1-\beta_0\cos\theta)^4}\right]\sin^2\theta$$

b) In a relativistic approx

θ small, $\gamma\theta$ finite (order 1)

$$\frac{1}{(1-\beta_0\cos\theta)} \simeq \frac{1}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)} = \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$

So returning to the boxed expression on the previous page

$$\left[\underbrace{\frac{1}{\beta\cos\theta}}_{1}\left(\frac{1}{(1-\beta\cos\theta)^4}-\underset{\text{small}}{1}\right)\right]\overbrace{\sin^2\theta}^{\simeq\theta}$$

$$= 2^4\gamma^6\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^4}$$

And

$$\frac{dW}{d\Omega} = \frac{q^2}{4\pi^2 c}\frac{\beta^2}{\Delta T}\frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^4}\gamma^6$$

Usind

$$d\Omega \simeq 2\pi\, \theta\, d\theta$$

$$\simeq \pi\, d\theta^2$$

$$dW = \frac{q^2}{4\pi^2 c} \frac{\beta^2}{\Delta T} \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^4} \gamma^4\, d(\gamma\theta)^2 \quad \checkmark$$

(c)

To find the total energy radiated we integrate over $(\gamma\theta)^2 \equiv \xi$

$$W = \int_0^\infty \text{Const} \frac{\xi}{(1+\xi)^4} d\xi$$

You can do this integral by integrating by parts once:

$$W = \int_0^\infty \text{Const}\ \xi\ d \frac{1}{(1+\xi)^3} \left(-\frac{1}{3}\right)$$

$$= \frac{1}{3} \int_0^\infty d\xi \frac{1}{(1+\xi)^3}$$

$$= \frac{1}{3} \frac{-1}{2} \frac{1}{(1+\xi)^2} \Bigg|_0^\infty = \frac{+1}{6}$$

So reinstating the constant:

$$\boxed{W = \frac{q^2}{4\pi c} \frac{\beta^2}{\Delta T} \frac{\gamma^4}{6}}$$

The relativistic larmour says

$$P(T) = \frac{dW}{dT} = \frac{q^2}{4\pi} \frac{2}{3c^3} \gamma^6 a_{\parallel}^2$$

Now $a_{\parallel}$ is constant $= c\beta/\Delta T$, but γ^6 is not constant. So we need to integrate:

$$I \equiv \int_0^{\Delta T} dT \frac{1}{(1-\beta^2(T))^3} = \int_0^{\Delta T} dT \, \gamma^6$$

Using, $\beta(T) = \beta_0 - \beta_0 \frac{T}{\Delta T}$, $d\beta = \frac{\beta_0}{\Delta T} dT$

We have

$$I = \frac{\Delta T}{\beta_0} \int_0^{\beta_0} d\beta \frac{1}{(1-\beta^2)^3}$$

The integral is dominated by a short time period near $\beta \simeq \beta_0$, writing $\beta = \beta_0 + \Delta\beta$

$$\frac{1}{(1-\beta^2)^3} \simeq \frac{1}{(1-\beta_0^2 - 2\beta_0\Delta\beta)^3} \simeq \frac{\gamma_0^6}{(1-2\gamma_0^2\Delta\beta)^3}$$

So

$$I \simeq \Delta T \int_{-\beta_0}^{0} d\Delta\beta \; \frac{\gamma_0^6}{(1-2\gamma_0^2\Delta\beta)^3}$$

$x = \gamma^2 \Delta\beta$

$$I \simeq \Delta T \gamma_0^4 \int_{-\infty}^{0} dx \; \frac{1}{(1-2x)^3}$$

$$I = \Delta T \, \gamma_0^4 \left(\frac{1}{4}\right)$$

So then

$$W = \int_0^{\Delta T} P(t)\, dT = \frac{q^2}{4\pi} \frac{2}{3} \frac{1}{c^3} \left(\frac{c\beta_0}{\Delta T}\right)^2 \overbrace{\frac{\gamma_0^4}{4} \Delta T}^{I}$$

$$\boxed{W = \frac{q^2}{4\pi c} \frac{\beta_0^2}{\Delta T} \cdot \frac{\gamma_0^4}{6}}$$