## Problem 1. Units

(a) The Gaussian unit system (or cgs) is often used in Quantum mechanics. Look it up on the internet, and write down the Coulomb and Biot Savart laws in this system of units.

(i) What is the relation between the electric and magnetic fields and the charges and currents of the Gaussian system and the Heavyside-Lorentz system (which we will use).

(ii) It is super-simple to convert from Gaussian to Heavyside Lorentz. Explain why and give the corresponding Gaussian expression for the electrostatic energy density $u_E = \frac{1}{2}E_{HL}^2$ .

Units without the "$4\pi$" in the Maxwell equations such as Heavyside Lorentz and SI (but with the $4\pi$ in the Coulomb law) are called "rationalized".

(b) Show that electric field and magnetic field (in Heavyside Lorentz) have units $\sqrt{\text{(force)/area}}$ or $\sqrt{\text{energy/volume}}$.

(c) A rule of thumb that you may need in the lab is that coaxial cable has a capacitance of 12 pF/foot. That is why cable length must be kept to a minimum in high speed electronics.

The order of magnitude of this result is set by $\epsilon_o = 8.85\,\text{pF/m}$. In the Heavyside-Lorentz system capacitance is still $Q_{HL} = C_{HL}V_{HL}$. Show that $C_{HL}$ has units of meters, and that

$$C_{MKS} = 8.85\,pF \left(\frac{C_{HL}}{\text{meters}}\right) \tag{1}$$

(d) The "impedance of the vacuum" is $Z_o = \sqrt{\mu_o/\epsilon_o} = 376\,\text{Ohms}$. This is why high frequency antennas will typically have a "radiation resistance" of this order of magnitude. As this problem will discuss, the unit of resistance is s/m for the Heavyside Lorentz system, and "the impedance of the vacuum" is $1/c$

In Heavyside-Lorentz units Ohm's law still reads, $\boldsymbol{j}_{HL} = \sigma_{HL}\boldsymbol{E}_{HL}$, where $\sigma_{HL}$ is the conductivity, and $\boldsymbol{j}$ is the current per area. Show that the conductivity in Heavyside-Lorentz has units $[\sigma_{HL}] = 1/\text{seconds}$ and that $\sigma_{MKS} = \sigma_{HL}\epsilon_o$. Then show that a wire of length $L$ and radius $R_o$ has resistance

$$R_{MKS} = 376\,\text{Ohms}\;(R_{HL}c) \tag{2}$$

$$= 376\,\text{Ohms}\left(\frac{Lc}{\pi R_o^2\sigma_{HL}}\right) \tag{3}$$

What is $\sigma_{HL}$ for copper?

For most metals $\sigma$ is so large that it competes with the speed of light. As we will see the relevant quantity is the magnetic diffusion coefficient $D \equiv c^2/\sigma$, which for copper is of order

$$\frac{c^2}{\sigma} \sim \frac{\text{cm}^2}{\text{(millisec)}}\,. \tag{4}$$

Units

a) $F = \dfrac{Q^2}{4\pi r^2}$

So $\boxed{[Q] = \sqrt{Nm^2}}$

$F = QE$ and thus $\boxed{[E] = \dfrac{[F]}{[Q]} = \sqrt{N/m^2}}$

b) $Q_{HL} = C_{HL} V_{HL}$

Now $[V_{HL}] = [E_{HL}] \cdot m$

$= \sqrt{\dfrac{N}{m^2}} \cdot m$

So

$\sqrt{Nm^2} = [C_{HL}] \sqrt{\dfrac{N}{m^2}} \cdot m$

$\boxed{\dfrac{1}{m} = [C_{HL}]}$

Using

$\dfrac{Q_{mks}}{\sqrt{\varepsilon_0}} = Q_{HL}$ and $\sqrt{\varepsilon_0}\, E_{mks} = E_{HL}$

So $Q_{mks} = (\varepsilon_0 C_{HL}) V_{mks}$

So

$$C_{mks} = 8{,}85\ pF\left(\frac{C_{HL}}{meter}\right)$$

c) So $\quad j_{HL} = \sigma_{HL} E_{HL}$

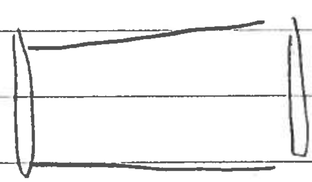
$$c\left(\frac{j_{HL}}{c}\right) = \sigma_{HL} E_{HL}$$

$$c\sqrt{\mu_0}\, j_{mks} = \sigma_{HL}\sqrt{\varepsilon_0}\, E_{mks}$$

$$j_{mks} = (\varepsilon_0 \sigma_{HL}) E_{mks}$$

So $\quad \sigma_{mks} = \sigma_{HL}\varepsilon_0$

And



$$I = j\pi R_0^2 = \sigma_{HL}\left(\frac{\pi R_0^2}{L}\right)\overbrace{LE}^{V}$$

$$\left(\frac{L}{\pi R_0^2 \sigma_{HL}}\right) I = V \Rightarrow \boxed{R_{HL} = \frac{L}{\pi R_0^2 \sigma_{HL}}}$$

or

$$\left(\frac{Lc}{\pi R_0^2 \sigma_{HL}}\right) \frac{I_{HL}}{c} = V_{HL}$$

$$\left(\frac{Lc}{\pi R_0^2 \sigma_{HL}}\right) \sqrt{\mu_0}\, I_{mks} = \sqrt{\varepsilon_0}\, V_{mks}$$

$$\left[\frac{Lc}{\pi R_0^2 \sigma_{HL}}\right] \sqrt{\frac{\mu_0}{\varepsilon}}\, I_{mks} = V_{mks}$$

$$\boxed{376 \text{ Ohms } (R_{HL} c) I_{mks} = V_{mks}}$$

## Problem 2. Vector Identities

(a) Use the epsilon tensor to prove the analog of "b(ac)-(ab)c" rule for curls

$$\nabla \times (\nabla \times \boldsymbol{V}) = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla^2 \boldsymbol{V} \tag{4}$$

Use this result, together with the Maxwell equations in the absence of charges and currents, to establish that $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the wave equation

$$\frac{1}{c^2}\partial_t^2 \boldsymbol{B} - \nabla^2 \boldsymbol{B} = 0 \tag{5}$$

$$\frac{1}{c^2}\partial_t^2 \boldsymbol{E} - \nabla^2 \boldsymbol{E} = 0 \tag{6}$$

(b) When differentating $1/r$ we write

$$\frac{1}{r} = \frac{1}{\sqrt{x^i x_i}} \tag{7}$$

with $\boldsymbol{x} = x^i \boldsymbol{e}_i$, and use results like

$$\partial_i x^j = \delta_i^j \qquad \partial_i x^i = \delta_i^i = d = 3 \tag{8}$$

where $d = 3$ is the number of spatial dimensions. (It is usually helps to write this as $d$ rather than 3 to get the algebra right). In this way, one finds that field due to a electric charge (monopole) is the familiar $\hat{\boldsymbol{r}}/r^2$

$$j\text{-th component of} \quad -\nabla(1/r) = \left(-\nabla\frac{1}{r}\right)_j = -\partial_j \frac{1}{\sqrt{x^i x_i}} = \frac{\frac{1}{2}(x^i\delta_{ji} + x_i\delta_j^i)}{\sqrt[3]{x^k x_k}} = \frac{x_j}{r^3} = \frac{(\hat{\boldsymbol{r}})_j}{r^2} \tag{9}$$

where $\hat{\boldsymbol{r}} \equiv \boldsymbol{n} = \boldsymbol{x}/r$.

Using tensor notation (*i.e.* indexed notation) show that

$$\nabla \times \frac{\hat{\boldsymbol{r}}}{r^2} = 0 \tag{10}$$

(c) Using the tensor notation (*i.e.* indexed notation) show that for constant vector $\boldsymbol{p}$ (and away from $\boldsymbol{r} = 0$) that

$$-\nabla\left(\frac{\boldsymbol{p}\cdot\boldsymbol{n}}{4\pi r^2}\right) = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} - \boldsymbol{p}}{4\pi r^3} \tag{11}$$

**Remark:** $\phi_{\text{dip}} = \boldsymbol{p}\cdot\boldsymbol{n}/(4\pi r^2)$ is the electrostatic potential due to an electric dipole $\boldsymbol{p}$, and Eq. (11) records the corresponding electric field. Notice the $1/r^3$ as opposed to $1/r^2$ for the monopole, and, taking $\boldsymbol{p}$ along the z-axis, notice how the electric field points at $\theta = 0$ (or $\boldsymbol{n} = \hat{\boldsymbol{z}}$) and $\theta = \pi/2$ (or $\boldsymbol{n} = \hat{\boldsymbol{x}}$). How could you derive this using the identities on the front cover of Jackson?

## Vector identities

a) $\nabla \times \nabla \times V$

Using

$$(\nabla \times \nabla \times V)^i = \varepsilon^{ijk}\, \partial_j\, \varepsilon_k{}^{lm}\, \partial_l V_m$$

$$= \varepsilon^{ijk}\, \varepsilon_k{}^{lm}\, \partial_j \partial_l V_m$$

$$= (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\, \partial_j \partial_l V_m$$

$$= \partial^i (\partial \cdot V) - \partial_j \partial^j V^i$$

$$= [\nabla(\nabla \cdot V) - \nabla^2 V]^i$$

Then

(1) $\nabla \cdot E = 0$

(2) $\nabla \times B = \frac{1}{c}\partial_t E$

(3) $\nabla \cdot B = 0$

(4) $-\nabla \times E = \frac{1}{c}\partial_t B$

Taking $\nabla \times$ (Eq. 2) gives after using Eq. 4

$$\nabla \times \nabla \times B = \frac{1}{c}\partial_t\left(-\frac{1}{c}\partial_t B\right)$$

Or using b(ac) −(ab)c rule

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}B - \nabla^2 B + \vec{\nabla}(\cancelto{0}{\nabla \cdot B}) = 0$$

or

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{B} = 0}$$

Similarly for $E$, taking $\nabla \times$ (Eq. 4) using Eq. 2 gives

$$-\nabla \times \nabla \times E = \frac{1}{c}\partial_t \frac{1}{c}\partial_t E$$

$$\boxed{\frac{1}{c^2}\partial_t^2 E - \nabla^2 E = 0}$$

b) Show

$$\nabla \times \frac{\vec{r}}{r^3} = 0$$

Using

$$\left(\nabla \times \frac{\hat{r}}{r^2}\right)^i = \varepsilon^{ijk} \partial_j \frac{r_k}{r^3}$$

$$= \varepsilon^{ijk} \left(\frac{\delta_{jk}}{r^3} - \frac{3}{\cancel{2}} \frac{r_k \cancel{2} r_j}{r^5}\right)$$

$$= \varepsilon^{ijk} \left(\frac{\delta_{jk}}{r^3} - 3 \frac{r_k r_j}{r^5}\right) = 0$$

c)

$$\left[-\nabla \left(\frac{\vec{p} \cdot \vec{r}}{4\pi r^3}\right)\right]_j = -\partial_j \left(\frac{p^l r_l}{4\pi r^3}\right)$$

$$= \left[-p^l \frac{\delta_{jl}}{r^3} + p^l r_l \left(\frac{+3}{\cancel{2}} \frac{\cancel{2} r_j}{r^5}\right)\right] \frac{1}{4\pi}$$

$$= \left[\frac{3(p \cdot n)\vec{n} - \vec{p}}{4\pi r^3}\right]_j$$

## Problem 3. Easy important application of Helmholtz theorems

(a) Using the source free Maxwell equations (*i.e.* those without $\rho$ and $\mathbf{j}$) and the Helmholtz theorems, explain why $\boldsymbol{E}$ and $\boldsymbol{B}$ can be written in terms of a scalar field $\Phi$ (the scalar potential) and a vector field $\boldsymbol{A}$ (the vector potential)

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{12}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\Phi \tag{13}$$

Thus two of the four Maxwell equations are trivially solved by introducing $\Phi$ and $\boldsymbol{A}$.

(b) Show that $\boldsymbol{A}$ and $\Phi$ are not unique, *i.e.*

$$A_i = (A_{\rm old})_i + \partial_i \Lambda(t, \boldsymbol{x}) \tag{14}$$

$$\Phi = (\Phi_{\rm old}) - \frac{1}{c}\partial_t \Lambda(t, \boldsymbol{x}) \tag{15}$$

gives the same $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Here $\Lambda(t, \boldsymbol{x})$ is any function. This change of fields is known as a gauge transformation of the gauge fields ($\Phi$, $\boldsymbol{A}$).

Helmholtz

(1) $\nabla \cdot B = 0 \implies B = \nabla \times \vec{A}$ Helmholtz

(2) $-\nabla \times E = \frac{1}{c}\partial_t B$

Then using (2) and $B = \nabla \times A$

$$-\nabla \times E + \frac{1}{c}\partial_t \nabla \times A = 0$$

$$-\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

So using Helmholtz, $\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\nabla \phi$, since

a scaler

the $\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$. So

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla \phi$$

b) If $A_i \to A_i + \partial_i \Lambda$

Then

$$B^j_{new} = \varepsilon^{jkl}\partial_k (A_l + \partial_l \Lambda)$$

$$= \underbrace{\varepsilon^{jkl}\partial_k A_l}_{B_{old}} + \underbrace{\varepsilon^{jkl}\partial_k \partial_l \Lambda}_{= 0}$$

The underlined term is zero since $\varepsilon^{jkl}$ is antisymmetric in $k, l$, while $\partial_k \partial_l \Lambda$ is symmetric in $k, l$. Thus

$$B^j_{new} = B^j_{old} + 0$$

Similarly:

$$(E_{new})_i = -\frac{1}{c}\partial_t A_i - \frac{1}{c}\partial_t \partial_i \Lambda - \left(\partial_i \phi - \frac{1}{c}\partial_t \partial_i \Lambda\right)$$

$$= -\frac{1}{c}\partial_t A_i - \partial_i \phi = E_{old,i}$$

## Problem 4. Tensor decomposition

(a) Consider a tensor $T^{ij}$, and define the symmetric and anti-symmetric components

$$T^{ij}_S = \tfrac{1}{2}\left(T^{ij} + T^{ji}\right) \tag{16}$$
$$T^{ij}_A = \tfrac{1}{2}\left(T^{ij} - T^{ji}\right) \tag{17}$$

so that $T^{ij} = T^{ij}_S + T^{ij}_A$. Show that the symmetric and anti-symmetric components don't mix under rotation

$$\underline{T_S}^{ij} = R^i{}_\ell R^j{}_m T^{\ell m}_S \tag{18}$$
$$\underline{T_A}^{ij} = R^i{}_\ell R^j{}_m T^{\ell m}_A \tag{19}$$

This means that I don't need to know $T_A$ if I want to find $\underline{T_S}$ in a rotated coordinate system.

**Remarks:** We say that the general rank two tensor is reducable to $T^{ij} = T^{ij}_S + T^{ij}_A$ into two tensors that dont mix under rotation

(b) You should recognize that an antisymmetric tensor is isomorphic to a vector

$$V_i \equiv \tfrac{1}{2}\epsilon_{ijk} T^{jk}_A \tag{20}$$

Explain qualitatively the identity $\epsilon^{ijk}\epsilon_{\ell mk} = \delta^i{}_\ell \delta^j{}_m - \delta^j{}_\ell \delta^i{}_m$ using $\epsilon^{ij3}\epsilon_{\ell m3}$ as an example, and use this to show

$$T^{ij}_A = \epsilon^{ijk} V_k \tag{21}$$

**Remark:** In matrix form this reads

$$T_A = \begin{pmatrix} 0 & V_z & -V_y \\ -V_z & 0 & V_x \\ V_y & -V_x & 0 \end{pmatrix} \tag{22}$$

(c) Using the Einstein summation convention, show that the trace of a symmetric tensor is rotationally invariant

$$\underline{T}^i{}_i \equiv T^i{}_i \tag{23}$$

and that

$$\mathring{T}^{ij}_S \equiv T^{ij} - \tfrac{1}{3}\delta^{ij} T^\ell_\ell \tag{24}$$

is traceless.

**Remark:** A symmetric tensor is therefore reducable to a symmetric traceless tensor and a scalar times $\delta^{ij}$.

$$T^{ij}_S = \mathring{T}^{ij}_S + \tfrac{1}{3}\delta^{ij} T^\ell_\ell \qquad \text{where} \qquad \mathring{T}^{ij}_S \equiv T^{ij}_S - \tfrac{1}{3} T^\ell{}_\ell \delta^{ij} \tag{25}$$

I don't need to know $T^\ell_\ell$ in order to compute $\underline{\mathring{T}^{ij}_S} = R^i{}_\ell R^j{}_m \mathring{T}^{\ell m}_S$

**Remarks:** The results of this problem show that a general second rank tensor is decomposable into irreducable components

$$T^{ij} = \mathring{T}_S^{ij} + \epsilon^{ijk} V_k + \tfrac{1}{3} T^\ell_\ell \delta^{ij} \tag{26}$$

$$= \frac{1}{2}\left(T^{ij} + T^{ji} - \tfrac{2}{3} T^\ell_\ell \delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}\epsilon_{k\ell m} T^{\ell m} + \frac{1}{3} T^\ell_\ell \delta^{ij} \tag{27}$$

No further reduction is possible. A general result is that a fully symmetric traceless tensor is irreducable.

When this result is applied to the product of two vectors it says

$$E^i B^j = \frac{1}{2}\left(E^i B^j + B^i E^j - \tfrac{2}{3}\boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij}\right) + \frac{1}{2}\epsilon^{ijk}(\boldsymbol{E}\times\boldsymbol{B})_k + \frac{1}{3}\boldsymbol{E}\cdot\boldsymbol{B}\delta^{ij} \tag{28}$$

which expresses the tensor product of two vectors as the sum of an irreducable (traceless and symmetric) tensor, a vector, and a scalar, $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

More physically it says that not all of $E_i B_j$ is really described by a tensor. Rather, part of $E_i B_j$ is described by the vector $\boldsymbol{E}\times\boldsymbol{B}$, and part is described by the scalar $\boldsymbol{E}\cdot\boldsymbol{B}$. It is for this reason that the tensors we work with in physics (*i.e.* the moment of inertia tensor, the quadrupole tensor, the maxwell stress tensor) are symmetric and traceless.

Problem 2

Using

a) $$\underline{T_S^{ij}} = \frac{1}{2}(\underline{T^{ij}} + \underline{T^{ji}})$$

$$= \frac{1}{2}\left(R^i{}_l R^j{}_m T^{lm} + R^j{}_l R^i{}_m T^{lm}\right)$$

But $l$ and $m$ are dummy labels (summed over)

So we are free to replace in the second them $m \to l$ $l \to m$

$$= \frac{1}{2}\left(R^i{}_l R^j{}_m T^{lm} + R^j{}_m R^i{}_l T^{ml}\right)$$

$$\boxed{\underline{T_S^{ij}} = R^i{}_l R^j{}_m \frac{1}{2}(T^{lm} + T^{ml})}$$

$$= R^i{}_l R^j{}_m T_S^{lm}$$

Similar steps go through for $T_A$

$$\boxed{\underline{T_A^{ij}} = R^i{}_l R^j{}_m \frac{1}{2}(T^{lm} - T^{ml})}$$

b) $\varepsilon^{ijk}\,\varepsilon_{lmk}$

Take $k=3$ the $ij = 1,2$ or $2,1$

Similarly $lm = 1,2$ or $2,1$

So either $i = l$ and $j = m$

$$\begin{array}{lcccc} ij & 1,2 & & 2,1 \\ & & \text{or} & \\ lm & 1,2 & & 2,1 \end{array}$$

<u>or</u>

$i = m$ and $j = l$

$$\begin{array}{lcccccc} ij & 1,2 & 2,1 & & 1,2 & 2,1 \\ & & & \text{or} & & \\ lm & 1,2 & 2,1 & & 1,2 & 2,1 \end{array}$$

leading to:

$$\varepsilon^{ijk}\,\varepsilon_{lmk} = \delta^i{}_l\,\delta^j{}_m - \delta^j{}_l\,\delta^i{}_m$$

So

$$T_A^{ij} = \varepsilon^{ijk}\,\overbrace{\tfrac{1}{2}\,\varepsilon_{klm}\,T^{lm}}^{\equiv V_k}$$

$$= \frac{1}{2}\left(\delta^i{}_l\,\delta^j{}_m - \delta^j{}_l\,\delta^i{}_m\right) T_A^{lm}$$

Leading to

$$T_A^{ij} = \frac{1}{2} T_A^{ij} - \frac{1}{2} \underbrace{T_A^{ji}}_{-T_A^{ij}}$$

$$T_A^{ij} = T_A^{ij} \quad \checkmark$$

c) $\underline{T_i^i} = \delta_{ij} T^{ij}$

$$= \delta_{ij} R^i{}_l R^j{}_m T^{lm}$$

$$= R^i{}_l R_{im}$$

$$= R^i{}_l (R^T)_{mi}$$

$$= R^T{}_{mi} R^i{}_l \; T^{lm}$$

$$= (R^T R)_{ml} T^{lm}$$

$$= \delta_{ml} T^{lm}$$

$$\underline{T^i{}_i} = T^l{}_l \quad \checkmark$$

Then

$$\overset{\circ}{T}{}^{ij} = T^{ij} - \frac{1}{3} \delta^{ij} T^l{}_l$$

$$\overset{\circ}{T}{}^i{}_i = T^i{}_i - \frac{1}{3} \underbrace{\delta^i{}_i}_{=3} T^l{}_l = T^i{}_i - T^l{}_l = 0$$

## Problem 5. 3d delta-functions

A delta-function in 3 dimensions $\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)$ is an infinitely narrow spike at $\boldsymbol{r}_o$ which satisfies

$$\int d^3\boldsymbol{r}\, \delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = 1 \tag{29}$$

In spherical coordinates, where the measure is

$$d^3\boldsymbol{r} = r^2 dr\, d(\cos\theta)\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi\,, \tag{30}$$

we must have

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = \frac{1}{r^2}\delta(r-r_o)\delta(\cos\theta-\cos\theta_o)\delta(\phi-\phi_o) = \frac{1}{r^2\sin\theta}\delta(r-r_o)\ \delta(\theta-\theta_o)\delta(\phi-\phi_o) \tag{31}$$

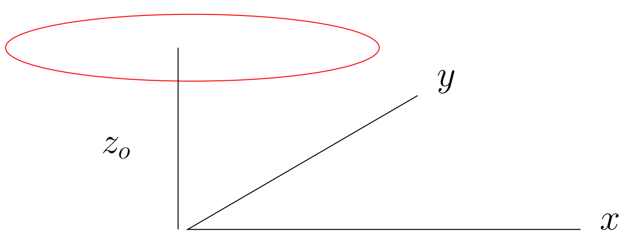
so that $\int d^3\boldsymbol{r}\, \delta^3(\boldsymbol{r}) = 1$. For a general curvlinear coordinate system

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o) = \frac{1}{\sqrt{g}} \prod_a \delta(u^a - u^a_o) \tag{32}$$

where $u^a_o$ are the coordinates of $\boldsymbol{r}_o$.

(a) What is formula $\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)$ for cylindrical coordinates?

(b) A uniform ring of charge $Q$ and radius $a$ sits at height $z_o$ above the $xy$ plane, and the plane of the ring is parallel to the $xy$ plane. Express the charge density $\rho(\boldsymbol{r})$ (charge per volume) in spherical coordinates using delta-functions. Check that the volume integral of $\rho(\boldsymbol{r})$ gives the total $Q$.



## Problem 5

a) $\delta^3(\vec{r}-\vec{r}_0) = \frac{1}{\rho}\delta(\rho-\rho_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)$

b) Then for a ring

$$\rho(r,\theta,\phi) = \frac{1}{r^2}\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\lambda a$$

where $r_0 \equiv \sqrt{a^2+z_0^2}$ $\quad \cos\theta_0 = \frac{z_0}{r_0}$

Then

$$Q = \int r^2\,dr\,d(\cos\theta)\,d\phi\left[\frac{1}{r^2}\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\right]\lambda a$$

$Q = \lambda a 2\pi$ $\quad$ where $\lambda$ = charge / length

## Problem 6. Coordinate systems and craziness

This problem discusses curvilinear coordinates and may well be too easy for you. You should read it though to help us collectively develop a common notation. For definiteness, take cylindrical coordinates

$$x = \rho\cos(\phi)\,, \tag{35}$$

$$y = \rho\sin(\phi)\,, \tag{36}$$

$$z = z\,, \tag{37}$$

though the results are easily generalized to spherical coordinates

$$x = r\sin\theta\,\cos(\phi), \tag{38}$$

$$y = r\sin\theta\,\sin(\phi), \tag{39}$$

$$z = r\cos\theta\,. \tag{40}$$

The position vector is $\boldsymbol{s} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$. The coordinate vectors $\mathbf{g}_\rho$, $\mathbf{g}_\phi$ and $\mathbf{g}_z$ are defined to point in the direction of increasing $\rho$, $\phi$ and $z$

$$\mathbf{g}_\rho \equiv \frac{\partial \boldsymbol{s}}{\partial \rho}\,, \qquad \mathbf{g}_\phi \equiv \frac{\partial \boldsymbol{s}}{\partial \phi}\,, \qquad \mathbf{g}_z \equiv \frac{\partial \boldsymbol{s}}{\partial z}\,, \tag{41}$$

so that displacement $\mathrm{d}\boldsymbol{s}$ is

$$\mathrm{d}\boldsymbol{s} = \mathrm{d}\rho\,\mathbf{g}_\rho + \mathrm{d}\phi\,\mathbf{g}_\phi + \mathrm{d}z\,\mathbf{g}_z \equiv \mathrm{d}u^a\mathbf{g}_a\,. \tag{42}$$

Here we have defined the coordinates and vectors generically with $u^a$ and $\mathbf{g}_a$ with indices $a, b, c\ldots$ drawn from the start of the alphabet

$$(u^1, u^2, u^3) \equiv (\rho, \phi, z)\,, \qquad \text{and} \qquad (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \equiv (\mathbf{g}_\rho, \mathbf{g}_\phi, \mathbf{g}_z)\,. \tag{43}$$

The vectors $\mathbf{g}_a$ are orthogonal in all the coordinate systems we will use, but they are not normalized. The squared displacement is then

$$\mathrm{d}s^2 \equiv \mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s} = \mathbf{g}_a\cdot\mathbf{g}_b\;\mathrm{d}u^a\mathrm{d}u^b \equiv g_{ab}\;\mathrm{d}u^a\mathrm{d}u^b = (\mathrm{d}\rho)^2 + \rho^2(\mathrm{d}\phi)^2 + (\mathrm{d}z)^2\,. \tag{44}$$

The factor of $\rho^2$ in $\rho^2(\mathrm{d}\phi)^2$ arises because $\mathbf{g}_\phi\cdot\mathbf{g}_\phi = \rho^2$ as you will show below. The off diagonal dot products (such as $\mathbf{g}_\phi\cdot\mathbf{g}_\rho = 0$) are zero for orthogonal coordinate systems, and the remaining diagonal dot products are unity. We have defined the metric tensor

$$g_{ab} \equiv \mathbf{g}_a\cdot\mathbf{g}_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{45}$$

and for orthogonal coordinate systems this matrix is diagonal and records the lengths of the chosen coordinate vectors.

The unit coordinate vectors that we normally use, $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_z$, are the normalized versions of $\mathbf{g}_\rho$, $\mathbf{g}_\phi$, and $\mathbf{g}_z$. Dividing by the lengths of each vector we have

$$\boldsymbol{e}_\rho \equiv \widehat{\boldsymbol{\rho}} = \mathbf{g}_\rho, \tag{46}$$

$$\boldsymbol{e}_\phi \equiv \widehat{\boldsymbol{\phi}} = \frac{\mathbf{g}_\phi}{\rho}, \tag{47}$$

$$\boldsymbol{e}_z \equiv \hat{\boldsymbol{z}} = \mathbf{g}_z. \tag{48}$$

Then a generic vector $\boldsymbol{V}$ is expanded as[1]

$$\boldsymbol{V} = V^a \boldsymbol{e}_a = V^\rho \boldsymbol{e}_\rho + V^\phi \boldsymbol{e}_\phi + V^z \boldsymbol{e}_z \,. \tag{49}$$

We have discussed cylindrical coordinates for definiteness. For a general orthogonal coordinate system[2], labeled by $u^1, u^2, u^3$, the coordinate vectors are defined as above

$$\mathbf{g}_a \equiv \frac{d\boldsymbol{s}}{du^a} \,, \tag{50}$$

and the squared length (also called the line-element) takes the generic form[3]

$$ds^2 = (h_a)^2 \, \mathrm{d}u^a \, \mathrm{d}u^a = (h_1)^2 \, (\mathrm{d}u^1)^2 + (h_2)^2 \, (\mathrm{d}u^2)^2 + (h_3)^2 \, (\mathrm{d}u^3)^2 \,. \tag{51}$$

where the scale factors are $(h_a)^2 = \mathbf{g}_a \cdot \mathbf{g}_a$ (no sum over a). The metric is clearly

$$g_{ab} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix} \tag{52}$$

The normalized unit vectors are

$$\boldsymbol{e}_a = \frac{\mathbf{g}_a}{h_a} \tag{53}$$

Then

$$g_{ab} = \mathbf{g}_a \cdot \mathbf{g}_b = h_a^2 \, \delta_{ab} \,, \qquad \text{and} \qquad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab} \,. \tag{54}$$

The volume of cell (the volume element) is

$$\mathrm{d}V = \sqrt{g} \, \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \,, \tag{55}$$

[1]In general relativity the symbol $V^a$ most often denotes an expansion of $\boldsymbol{V}$ in terms of the coordinate vectors $\mathbf{g}_a$, i.e. $\boldsymbol{V} = V^a \mathbf{g}_a$. An expansion of $\boldsymbol{V}$ in the normalized coordinate vectors would then be denoted, $\boldsymbol{V} = V^{\hat{a}} \boldsymbol{e}_{\hat{a}}$, with an extra hat. This convention is almost never followed in electricity and magnetism. So what we (and everyone else) call $V^\phi$ (the component of the vector in the $\phi$ direction) is what a pedantic general relativist would call $V^{\hat{\phi}}$.

[2]For non-orthogonal coordinate systems a nice elementary book is James G. Simmonds *A Brief on Tensor Analysis*.

[3]Compare this to Eq. (44). $(\mathrm{d}u^3)^2$ means the square of the change in coordinate number three, $(\mathrm{d}z)^2$ above. The scale factors $h_a$ will always be written with a lower index. Here and below (when confusion does not arise) repeated indices involving $h_a$ are not summed over, but simply multiply the term in the sum, e.g. $d\boldsymbol{s} = h_a \, du^a \boldsymbol{e}_a = \mathrm{d}\rho \, \boldsymbol{e}_\rho + \rho \, \mathrm{d}\phi \, \boldsymbol{e}_\phi + \mathrm{d}z \, \boldsymbol{e}_z$

where $g = \det g_{ab}$ is the determinant of the metric matrix

$$\sqrt{g} = \sqrt{\det g_{ab}} = h_1 h_1 h_3 \tag{56}$$

for orthogonal coordinate systems.

All of the tensor analysis done in class which does not involve differentiation (i.e. tensor decomposition, cross products etc) goes through without change. As long as no derivatives are involved, the change of coordinates is just an orthonormal change of basis at each point in space from the $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ vectors (labeled generically as $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ with indices $i, j, k, \ldots$) to the $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$, $\boldsymbol{e}_z$ vectors (labeled generically as $\boldsymbol{e}_a$ and $\boldsymbol{e}_b$ with indices $a, b, c \ldots$). The physical vector $\boldsymbol{V}$ or tensor $\boldsymbol{T}$ is the same in both coordinate systems.

$$\boldsymbol{V} = V^i \boldsymbol{e}_i = V^a \boldsymbol{e}_a, \qquad \boldsymbol{T} = T^{ij} \boldsymbol{e}_i \boldsymbol{e}_j = T^{ab} \boldsymbol{e}_a \boldsymbol{e}_b. \tag{57}$$

One simply needs to express the curvilinear basis vectors in terms of the Cartesian ones as you will do below.

The curvilinear differential operators are more complicated, because the basis vectors are functions of the spatial coordinates.

(a) The gradient of scalar is

$$\nabla\Phi = \boldsymbol{e}_a \frac{1}{h_a} \frac{\partial \Phi}{\partial u^a} = \frac{\boldsymbol{e}_1}{h_1} \frac{\partial \Phi}{\partial u^1} + \frac{\boldsymbol{e}_2}{h_2} \frac{\partial \Phi}{\partial u^2} + \frac{\boldsymbol{e}_3}{h_3} \frac{\partial \Phi}{\partial u^3} \tag{58}$$

(b) The divergence

$$\nabla \cdot \boldsymbol{V} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} V^a / h^a)}{\partial u^a} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 V^1)}{\partial u^1} + \frac{\partial (h_1 h_3 V^2)}{\partial u^2} + \frac{\partial (h_1 h_2 V^3)}{\partial u^3} \right] \tag{59}$$

(c) The curl

$$(\nabla \times \boldsymbol{V}) = \boldsymbol{e}_a \, \epsilon^{abc} \frac{1}{h_b h_c} \partial_b (h_c V^c) \tag{60}$$

Or

$$\begin{aligned}(\nabla \times \boldsymbol{V}) = \frac{\boldsymbol{e}_1}{h_2 h_3} \left[ \frac{\partial (h_3 V^3)}{\partial u^2} - \frac{\partial (h_2 V^2)}{\partial u^3} \right] + \frac{\boldsymbol{e}_2}{h_1 h_3} \left[ \frac{\partial (h_1 V^1)}{\partial u^3} - \frac{\partial (h_3 V^3)}{\partial u^2} \right] \\ + \frac{\boldsymbol{e}_3}{h_1 h_2} \left[ \frac{\partial (h_2 V^2)}{\partial u^1} - \frac{\partial (h_1 V^1)}{\partial u^2} \right]\end{aligned} \tag{61}$$

(d) The Laplacian follows from the divergence and gradient

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^a} \left[ \frac{\sqrt{g}}{(h_a)^2} \frac{\partial \Phi}{\partial u^a} \right] \tag{62}$$

The Laplacian is often expressed in term of the inverse metric, i.e. the matrix inverse of the metric $g_{ab}$

$$g_{ab} (g^{-1})^{bc} = \delta_a^c \tag{63}$$

The inverse metric is

$$(g^{-1})^{ab} = \begin{pmatrix} \frac{1}{(h_1)^2} & 0 & 0 \\ 0 & \frac{1}{(h_2)^2} & 0 \\ 0 & 0 & \frac{1}{(h_3)^2} \end{pmatrix} \tag{64}$$

Thus the Laplacian is

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^a} \left( \sqrt{g} (g^{-1})^{ab} \frac{\partial \Phi}{\partial u^b} \right) \tag{65}$$

which provides a simple way to remember the Laplacian in spherical coordinates.

Now very briefly (showing as much or as little work as you care to) answer the following questions:

(a) Using algebraic means (i.e. without drawing a little picture) and starting from the definitions in Eq. (41), express $\mathbf{g}_\rho$, $\mathbf{g}_\phi$, and $\mathbf{g}_z$ and $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ in Cartesian components, $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. Prove Eq. (44) algebraically. Confirm to yourself that this algebra agrees with the visual picture that you have of these vectors.

(b) One of the things that I see all the time (including several times on last week's comps) that makes me completely crazy is the following tragically flawed logic

$$\frac{d}{d\phi} \left[ \cos(\phi)\, \widehat{\boldsymbol{\rho}} \right] = -\sin\phi\, \widehat{\boldsymbol{\rho}} \qquad \text{crazily wrong!} \tag{66}$$

What is the correct answers to this derivative? Express your answer in terms of $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$.

(c) Similarly, every year a few tragically flawed exams have expressions like

$$\int_0^{2\pi} d\phi \cos\phi\, \widehat{\boldsymbol{\rho}} = 0 \qquad \text{crazily wrong!} \tag{67}$$

What is the correct answer to this integral?

(d) Determine $\mathbf{g}_r$, $\mathbf{g}_\theta$, $\mathbf{g}_\phi$, the metric $g_{ab}$ and line elements $\mathrm{d}s^2$ for spherical coordinates. Also determine the volume $\mathrm{d}V$ and Laplacian in spherical coordinates.

(e) (Zangwill) Express

$$\frac{\partial \widehat{\boldsymbol{r}}}{\partial \theta} \qquad \frac{\partial \widehat{\boldsymbol{r}}}{\partial \phi} \qquad \frac{\partial \widehat{\boldsymbol{\theta}}}{\partial \theta} \qquad \frac{\partial \widehat{\boldsymbol{\theta}}}{\partial \phi} \qquad \frac{\partial \widehat{\boldsymbol{\phi}}}{\partial \theta} \qquad \frac{\partial \widehat{\boldsymbol{\phi}}}{\partial \phi} \tag{68}$$

in terms of $\hat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\phi}}$. Can you visualize these results?

# Coordinate Systems

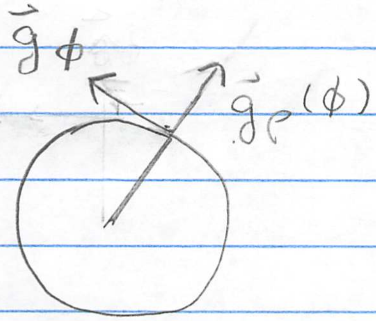
a) The coordinates are $\vec{s} = x\hat{x} + y\hat{y} + z\hat{z}$

or $\vec{s} = (x, y, z) = (\rho\cos\phi, \rho\sin\phi, z)$

① $\vec{g}_\rho = \frac{d\vec{s}}{d\rho} = (\cos\phi, \sin\phi, 0)$

② $\vec{g}_\phi = \frac{d\vec{s}}{d\phi} = \rho(-\sin\phi, \cos\phi, 0) \leftarrow |\vec{g}_\phi| = \rho$

③ $\vec{g}_z = \frac{d\vec{s}}{dz} = (0, 0, 1)$




So

$$\hat{\rho} = \vec{g}_\rho = (\cos\phi, \sin\phi, 0)$$

$$\hat{\phi} = \frac{\vec{g}_\phi}{|\vec{g}_\phi|} = \frac{\vec{g}_\phi}{\rho} = (-\sin\phi, \cos\phi, 0)$$

$$\hat{z} = (0, 0, 1)$$

i.e.

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

b) The point is that $\hat{\rho}$ is a function of angle:

$$\frac{\partial \hat{\rho}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) = \hat{\phi}$$

So

$$\frac{d}{d\phi}\left(\cos\phi\, \hat{\rho}\right) = -\sin\phi\, \hat{\rho} + \cos\phi\, \hat{\phi}$$

c) Similarly

$$\vec{I} = \int_0^{2\pi} d\phi \cos\phi \left(\cos\phi\, \hat{x} + \sin\phi\, \hat{y}\right)$$

$$= 2\pi \langle \cos^2\phi \rangle\, \hat{x}$$

$$= 2\pi\, \frac{1}{2}\, \hat{x} = \pi \hat{x}$$

We use the result:

$$\langle \sin^2 \rangle = \langle \cos^2 \rangle = 1/2$$
$$\langle \sin \cos \rangle = 0$$

d) For spherical coords:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\vec{g}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$$

$$\vec{g}_\theta = r\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\vec{g}_\phi = r\sin\theta\,(-\sin\phi, \cos\phi, 0)$$

Then

$$g_{ab} = \begin{pmatrix} \vec{g}_r \cdot \vec{g}_r & & \\ & \vec{g}_\theta \cdot \vec{g}_\theta & \\ & & \vec{g}_\phi \cdot \vec{g}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r^2} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right)$$

e) $$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} = \frac{\partial}{\partial \theta} (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

Similarly

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta \, \hat{\phi},$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta \, \hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}$$

## Problem 6. Fourier Transforms of the Coulomb Potential

The fourier transfrom takes a function in coordinate space and represents in momentum space[1]

$$F(k) = \int_{-\infty}^{\infty} dx \left[e^{-ikx}\right] f(x) \tag{33}$$

The inverse transformation repesents a function as a sum of plane waves

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikx}\right] F(k) \tag{34}$$

The Fourier transform generalizes the concept of a fourier series to non-periodic, but square integrable functions – *i.e.* $\int dx|f(x)|^2$ should converge.

The Fourier transform of a 3D function $\boldsymbol{r} = (x, y, z)$ is:

$$F(\boldsymbol{k}) = \int d^3\boldsymbol{r} \,\left[e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] F(\boldsymbol{r}) \tag{35}$$

$$F(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \,\left[e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] F(\boldsymbol{k}) \tag{36}$$

To do this problem you will need to know (as dicussed in class) that the integral of a pure phase $e^{ikx}$ is proportional to a delta-fcn. In $3D$ we have

$$\delta^3(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{37}$$

$$(2\pi)^3\delta^3(\boldsymbol{k}) = \int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{38}$$

(a) Use tensors notation to show that the Fourier transform of $\nabla F(\boldsymbol{r})$ is

$$i\boldsymbol{k}F(\boldsymbol{k})\,, \tag{39}$$

and that the Fourier transform of the curl of a vector vector field $\boldsymbol{F}(\boldsymbol{r})$ is $\nabla \times \boldsymbol{F}(\boldsymbol{r})$ is

$$i\boldsymbol{k} \times \boldsymbol{F}(\boldsymbol{k}) \tag{40}$$

(b) The genral rule is to replace $\nabla \to i\boldsymbol{k}$. What is the Fourier transform of $\nabla^2 F(\boldsymbol{r})$

(c) Prove the Convolution Theorem, *i.e.* the Fourier Transform of a product is a convolution

$$\int d^3\boldsymbol{r}\, e^{-i\Delta\boldsymbol{k}\cdot\boldsymbol{r}}\, |F(\boldsymbol{r})|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} F(\boldsymbol{k})F^*(\boldsymbol{k} - \Delta\boldsymbol{k}) \tag{41}$$

making liberal use of the completeness integrals

$$\int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = (2\pi)^3\delta^3(\boldsymbol{k}) \tag{42}$$

[1] The notation of putting $e^{ikx}$ in square brackets is not standard, but I have used it in the notes to highlight the similarity between this expansion and other eigenfunction expansions.

**Remark:** Setting $\Delta\boldsymbol{k} = 0$ we recover Parseval's Theorem

$$\int d^3r|F(\boldsymbol{r})|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}|F(\boldsymbol{k})|^2 \tag{43}$$

**Remark:** This is often used in reverse, the fourier transform of a convolution is a product of the fourier transforms

$$\text{F.T. of } \int d^3\boldsymbol{r}_o\, F(\boldsymbol{r}_o)\, G(\boldsymbol{r} - \boldsymbol{r}_o) \; = F(\boldsymbol{k})G(\boldsymbol{k}) \tag{44}$$

(d) The Fourier transform of the Coulomb potential is difficult (try it and find out why!). This is because $1/(4\pi r)$ is not in the space of square integrable functions (Why?). Thus, we will consider the Fourier transform of $1/(4\pi r)$ to be the limit as $m \to 0$ of the Fourier transform of a screened Coulomb potential known as the Yukawa potential

$$\Phi(\boldsymbol{x}) = \frac{e^{-m|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|} \tag{45}$$

The Yukawa potential is square integrable. Show that the Fourier transform of the Yukawa potential is

$$\Phi(\boldsymbol{k}) = \frac{1}{k^2 + m^2} \tag{46}$$

with $k = \sqrt{\boldsymbol{k}^2}$. Thus, we conclude with $m \to 0$ that

$$\int d^3\boldsymbol{x}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \frac{1}{4\pi r} = \frac{1}{k^2} \tag{47}$$

Note that the inverse transform can be computed by direct integration

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_o)}}{k^2} \tag{48}$$

(e) In electrostatics the electric field is the negative gradient of the potential, $\boldsymbol{E} = -\nabla\Phi$. From $\nabla \cdot \boldsymbol{E} = \rho$, we derive the Poisson equation $-\nabla^2\Phi = \rho$. For a unit charge at the origin, the coulomb potential, $1/(4\pi r)$, satisfies

$$-\nabla^2\Phi = \delta^3(\boldsymbol{r}) \tag{49}$$

Deduce Eq. (47) by fourier transforming this equation.

Fourier Transforms

a)

$$F(x) = \int \frac{d^3k}{(2\pi)} e^{ik\cdot x} F(k) \qquad \int_k = \int \frac{d^3k}{(2\pi)^3}$$

$$\partial_i F(x) = \int_k e^{ik\cdot x} k_i F(k)$$

So

$$\boxed{\partial_i F(x) \longleftrightarrow k_i F(k)}$$

Then,

$$(\nabla \times F)^i = \varepsilon^{ijk} \partial_j F_k = \int_k \varepsilon^{ijk} \partial_j e^{ik\cdot x} F_k(\vec{k})$$

$$= \int_k e^{ikx} \varepsilon^{ijk} k_j F_k(k)$$

$$(\nabla \times F)^i = \int_k e^{ikx} (\vec{k} \times \vec{F}(k))^i$$

b) $\nabla^2 F \longleftrightarrow -k^2 F(k)$

Notation:

c) $\int_r \equiv \int d^3r$, and note

$\int_k \equiv \int \frac{d^3k}{(2\pi)^3}$

Then

$$LHS = \int_r e^{-i\Delta k r} \underbrace{\int_k e^{ik\cdot r} F(k)}_{= F(r)} \underbrace{\int_{k'} e^{-ik'r} F^*(k')}_{= F^*(r)}$$

$$= \int_{kk'r} F(k) F^*(k') e^{-i(\Delta k - k + k')r}$$

$$= \int_{kk'} F(k) F^*(k') (2\pi)^3 \delta^3(\Delta k - k + k')$$

$$\boxed{LHS = \int_k F(k) F^*(k - \Delta k) = RHS}$$

d) To compute the FT of yukawa, it is best to use spherical coords

$$F(\vec{k}) = \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \frac{e^{-mr}}{4\pi r}$$

Taking $\vec{k}$ along the z-axis (this choice is arbitrary since we can always rotate our $\vec{r}$ coordinates)

$$\vec{k}\cdot\vec{r} = kr\cos\theta$$

Then

$$F(\vec{k}) = \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) \int d\phi \; e^{-ikr\cos\theta} \frac{e^{-mr}}{4\pi r}$$

$$= \int_0^\infty r^2 dr \, \frac{e^{-mr}}{4\pi r} \left[ \frac{e^{-ikr\cos\theta}}{-ikr} \right]_{-1}^{1} \cdot 2\pi$$

$$F(r) = \frac{1}{4\pi} \cdot 2\pi \int_0^\infty dr \, e^{-mr} \, \frac{2\sin kr}{k}$$

$$= \frac{1}{k^2} \int_0^\infty du \, e^{-um/k} \sin u \qquad u \equiv kr$$

So:

$$\boxed{F(k) = \frac{1}{k^2} \frac{1}{\left(1 + \frac{m^2}{k^2}\right)} = \frac{1}{k^2+m^2} \xrightarrow{m\to 0} \frac{1}{k^2}}$$

e)

Fourier transforming eq. e, $-\nabla^2 \Phi = \delta^3(r)$

$$k^2 \bar{\Phi}(k) = 1$$

$$\boxed{\bar{\Phi}(k) = \frac{1}{k^2}}$$