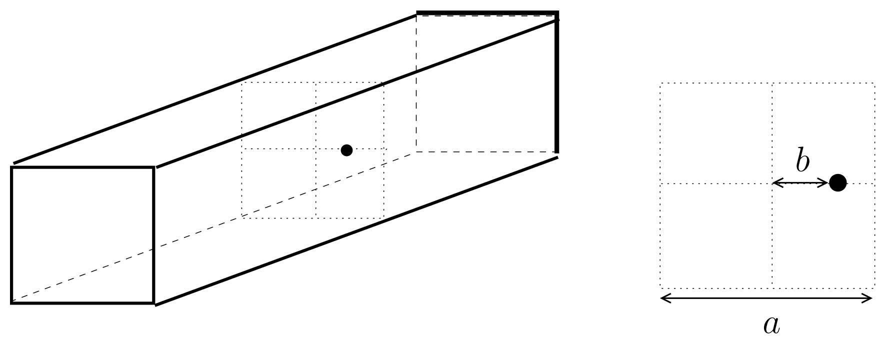
Do not hand in optional parts!

Problem 1. A point charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



(a) (Optional) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x x)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \left\{\cos(u) \text{ or } \sin(u)\right. \tag{1}$$

for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

(b) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential. You will need an infinite number of image charges of both sign.

(c) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b). You should find the form

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \frac{4q}{a^2} \sum_n \sum_m X_n(x) X_n(b)\, Y_m(y) Y_m(0) \frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{2}$$

where $X_n(x) = \Phi_n(k_n x)$ and $Y_m(y) = \Phi_m(k_m y)$ and $\kappa_{n,m} = \sqrt{k_n^2 + k_m^2}$.

(d) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$. You should find that the force per area on the bottom plate (far from the charge) is

$$\frac{F^y}{A} \simeq \frac{q^2}{a^4} \cos^2(\pi x/a) \cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{3}$$

Solution

1. The Laplace equation is

$$-\nabla^2\varphi = 0 \tag{2}$$

Separating variables with $\varphi = X(x)Y(y)Z(z)$ we must have

$$-\frac{d^2X}{dx^2} = k_x^2 X \tag{3a}$$

$$-\frac{d^2Y}{dy^2} = k_y^2 Y \tag{3b}$$

$$-\frac{d^2Z}{dz^2} = k_z^2 Z \tag{3c}$$

In order to satisfy Eq. (2), the separation constants satisfy

$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{4}$$

and thus

$$\frac{d^2Z}{dz^2} = \kappa^2 Z \qquad \text{with} \qquad \kappa = \sqrt{k_x^2 + k_y^2} \tag{5}$$

The solutions to Eq. (3a) may be either sines or cosines

$$X(x) = \Phi(k_x x)\,, \tag{6}$$

with k_x at this point still arbitrary. In order to satisfy the boundary conditions $X(\pm a/2) = 0$, we require for the cosine functions that

$$k_x a/2 = (n + \tfrac{1}{2})\pi\,. \tag{7}$$

Similarly, for the sin functions

$$k_x a/2 = n\pi\,. \tag{8}$$

Thus, the general form is

$$X_n(x) = \Phi_n(k_n x) \qquad n = 0, 1, \ldots, \tag{9}$$

with $k_n = (n+1)\pi/a$ and

$$\Phi_n(u) = \begin{cases} \cos(u) & \text{n even} \\ \sin(u) & \text{n odd} \end{cases}\,. \tag{10}$$

The $Y(y)$ direction follows by analogy

$$Y_m(y) = \Phi_m(k_m x) \qquad m = 0, 1, \ldots, \tag{11}$$

with $k_m = (m+1)\pi/a$. The solutions to the z direction are

$$Z(z) = e^{\pm\kappa z} \qquad \kappa = \sqrt{k_n^2 + k_m^2}\,. \tag{12}$$

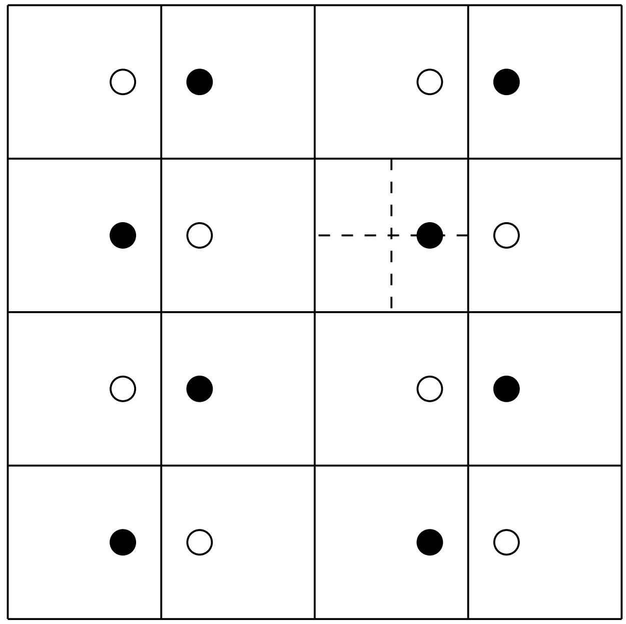


Figure 1: Arrangement of image charges. The black circles idicate plus charges, while the white circles indicate negative charges. The origin of the coordinate system is indicated by the dashed lines. The real charge is displaced by a distance b from the origin.

2. The image charges may be placed in a rectangular lattice as shown below. Their are four types of charges with coordinates

$$\boldsymbol{r}_1(n,m) =(b+2na)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{13}$$
$$\boldsymbol{r}_2(n,m) =((2n+1)a-b)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{14}$$
$$\boldsymbol{r}_3(n,m) =(b+2na)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{15}$$
$$\boldsymbol{r}_4(n,m) =((2n+1)a-b)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{16}$$

where n, m are integers. Then the potential is

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi}\sum_{n,m=0}^{\infty} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_2(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_3(n,m)|} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_4(n,m)|} \tag{17}$$

3. For the potential at $\boldsymbol{r}$ due to a point charge at $\boldsymbol{r}_o = (b, 0, 0)$, we expand the potential as

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) = \left(\frac{2}{a}\right)^2 \sum_{n,m=0}^{\infty,\infty} X_n(x)X_n(b)\, Y_m(y)Y_m(0)\, g_{n,m}(z) \tag{18}$$

and substitute into the Poisson equation

$$-\nabla^2\varphi(\boldsymbol{r};\boldsymbol{r}_o) = q\delta(x-b)\delta(y)\delta(z)\,. \tag{19}$$

The leading factors $(2/a)^2$ arise from the fact that we have not normalized the eigenfunctions X and Y

$$\int_{-a/2}^{a/2} dx\, X_n(x) X_{n'}(x) = \frac{a}{2}\, \delta_{n,n'} \tag{20}$$

$$\int_{-a/2}^{a/2} dy\, Y_m(y) Y_{m'}(y) = \frac{a}{2}\, \delta_{m,m'} \tag{21}$$

If $g_{n,m}(z)$ satisfies

$$\left(k_n^2 + k_m^2 - \frac{\partial^2}{\partial z^2} \right) g_{n,m}(z) = q\delta(z)\,, \tag{22}$$

then using the completeness relation

$$\frac{2}{a} \sum_n X_n(x) X_n(x_o) = \delta(x - x_o) \tag{23}$$

$$\frac{2}{a} \sum_m Y_m(x) Y_m(x_o) = \delta(y - y_o) \tag{24}$$

it is not difficult to show that Eq. (19) is satisfied.

The solution to Eq. (22) is

$$g_{n,m}(z) = \begin{cases} A e^{-\kappa_{n,m} z} & z > 0 \\ A e^{\kappa_{n,m} z} & z < 0 \end{cases} \tag{25}$$

Integrating across the δ-fcn in Eq. (22) we have

$$-\left.\frac{dg}{dz}\right|_{z=0^+} + \left.\frac{dg}{dz}\right|_{0^-} = q \tag{26}$$

With this requirement $A = \frac{q}{2\kappa_{n,m}}$ and

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \frac{4q}{a^2} \sum_{n,m=0}^{\infty,\infty} X_n(x) X_n(b)\, Y_m(y) Y_m(0) \frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{27}$$

4. At asymptotic distances the terms with the smallest $\kappa_{n,m}$ dominate the sum. We then have only the contribution from $n = m = 0$ mode, and

$$\kappa_{0,0} = \sqrt{2}\pi/a\,. \tag{28}$$

The potential reads

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) \simeq \frac{4q}{a^2} \cos(\pi x/a) \cos(\pi b/a)\, \cos(\pi y/a) \frac{e^{-\kappa_{0,0}|z|}}{2\kappa_{0,0}} \tag{29}$$

or

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) \simeq \frac{\sqrt{2}q}{\pi a} cos(\pi x/a)\cos(\pi b/a)\ \cos(\pi y/a) e^{-\sqrt{2}\pi|z|/a} \tag{30}$$

Let us calculate the charge density on the bottom plate

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{E} = -\partial_y \phi\big|_{y=-a/2}\ , \tag{31}$$

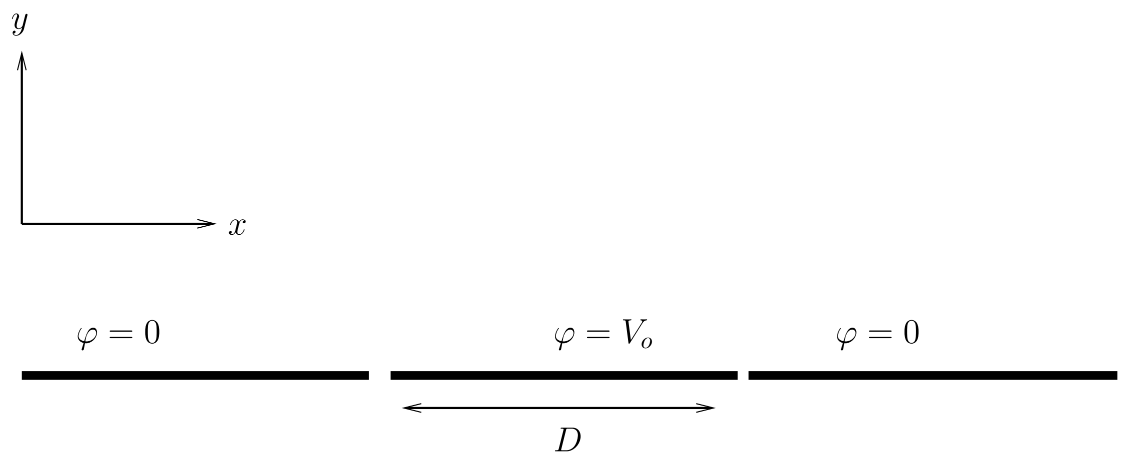
$$= -\frac{\sqrt{2}q}{a^2}\cos(\pi x/a)\cos(\pi b/a)\, e^{-\sqrt{2}\pi|z|/a}\,. \tag{32}$$

Finally, the force per area on the bottom plate is

$$\frac{F^y}{A} = \frac{\sigma^2}{2}\ , \tag{33}$$

$$= \frac{q^2}{a^4}\cos^2(\pi x/a)\cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{34}$$

The direction of the force is into the tube. The other walls of the tube have the same force per area.



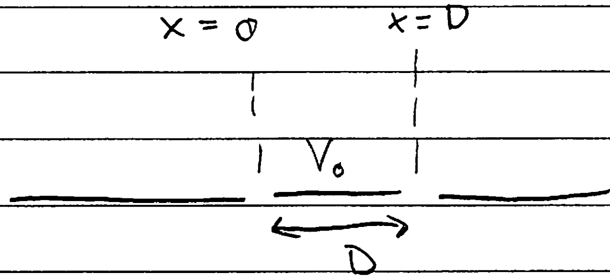
Problem 1. Potential from a strip.

An infinite conducting strip of width D (between $0 < x < D$) is maintained at potential V_o, while on either side of the strip are grounded conducting planes. The strip and the planes are separated by a tiny gap as shown below.

(a) Following a similar example given in class, determine the potential everywhere in the upper half plane $y > 0$.

(b) Determine the surface charge density on the strip and on the grounded planes, and make a graph.

Problem 5



$$\varphi(\vec{r}) = -\int dS\, \vec{n}_0 \cdot \vec{\nabla}_{r_0} G(r, r_0)\, \varphi(r_0)$$

$$G = -\frac{1}{2\pi} \log|\vec{r} - \vec{r}_0| + \frac{1}{2\pi} \log|\vec{r} - \vec{r}_{0I}|$$

$$\vec{r} = (x, y)$$

$$\vec{r}_0 = (x_0, y_0) \qquad \vec{r}_{0I} = (x_0, -y_0)$$

$$-\vec{n}_0 \cdot \vec{\nabla}_{r_0} G = +\frac{\partial G(\vec{r}, \vec{r}_0)}{\partial y_0}$$

$$\left.\frac{\partial G}{\partial y_0}\right|_{y_0 \to 0} = \frac{y}{\pi((x-x_0)^2 + y^2)}$$

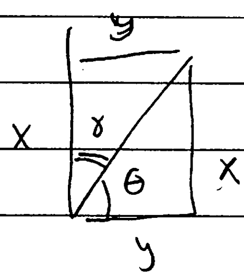
So

$$\varphi(\vec{r}) = \int_0^D dx_0 \frac{y}{\pi((x-x_0)^2 + y^2)} V_0$$

$$\varphi(r) = -\frac{V_0}{\pi}\, \operatorname{atan}\left(\frac{(x-x_0)}{y}\right)\Bigg|_0^D$$

$$= -\frac{V_0}{\pi}\, \operatorname{atan}\frac{(x-D)}{y} + \frac{V_0}{\pi}\, \operatorname{atan}\left(\frac{x}{y}\right)$$

Using



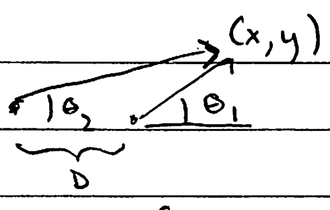
$$\theta = \operatorname{atan} x/y$$

$$\gamma = \operatorname{atan} y/x = \pi/2 - \theta$$

$$\operatorname{atan}\left(\frac{x}{y}\right) = \theta = \pi/2 - \operatorname{atan}(y/x)$$

$$\boxed{\varphi(r) = \frac{V_0}{\pi}\left[\operatorname{atan}\left(\frac{y}{x-D}\right) - \operatorname{atan}\left(\frac{y}{x}\right)\right]}$$

So check:



$$\theta_1 = \operatorname{atan} y/(x-D)$$

$$\theta_2 = \operatorname{atan} y/x$$

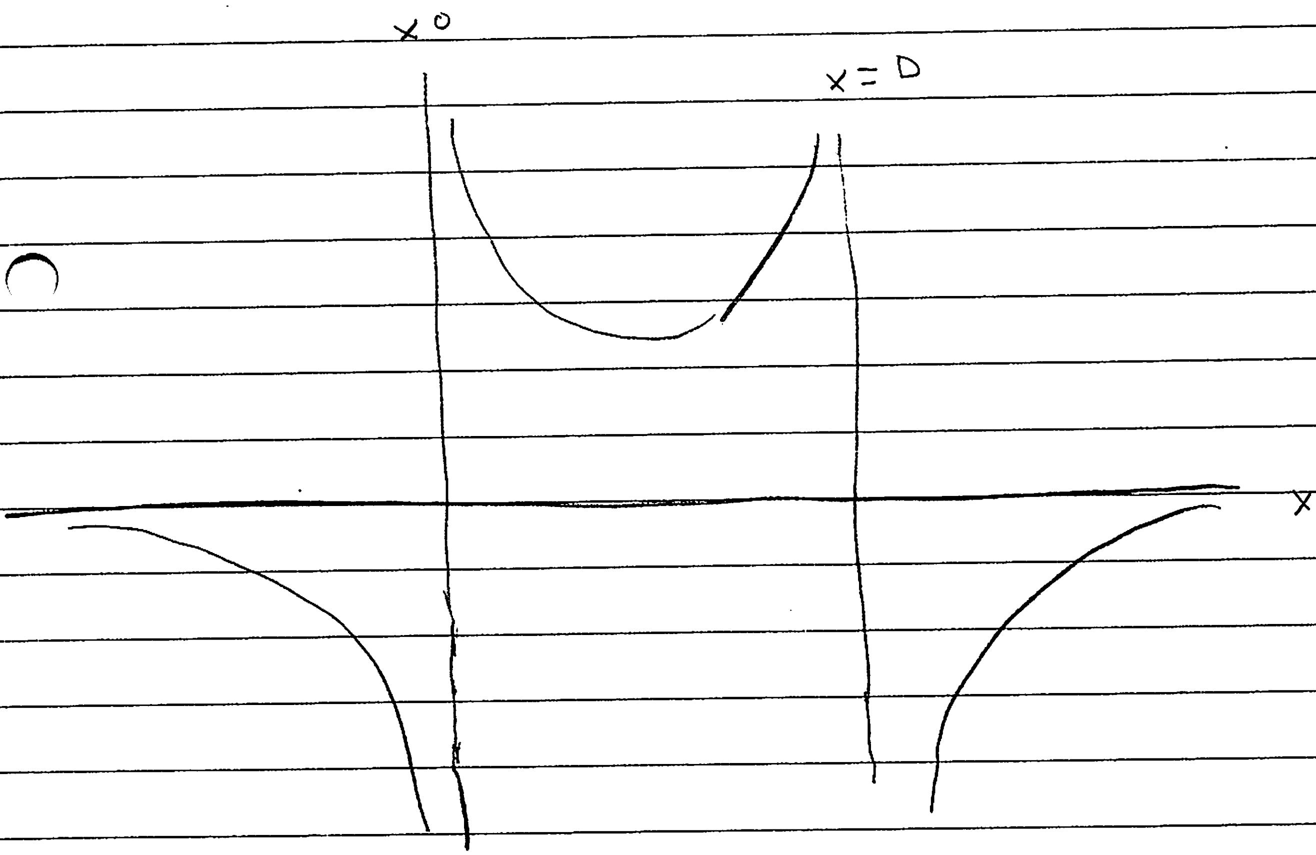
$$\varphi(r) = \frac{V_0}{\pi}\left[\theta_1 - \theta_2\right]$$

which clearly satisfies all B.C.

The surface charge:

$$E_n = -\frac{\partial \varphi}{\partial y}\bigg|_{y=0} = \frac{-V_o}{\pi(x-D)} + \frac{V_o}{\pi x}$$

Plot:



Problem 2. The electric stress tensor

Recall that the stress tensor is the force per area. The force per volume f^j is (minus) the divergence of the stress tensor (see class notes)

$$f^j = -\partial_i T^{ij} \tag{1}$$

This follows from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0 \tag{2}$$

where g^j is the momentum per volume, and the basic notion that the force is the time derivative of the momentum.

The force per volume in electrostatics is

$$f^j = \rho E^j \tag{3}$$

This form must be the divergence of something. As you will show in this excercise

$$\rho E^j = -\partial_i T_E^{ij} \tag{4}$$

where

$$T_E^{ij} \equiv -E^i E^j + \tfrac{1}{2} E^2 \delta^{ij} \tag{5}$$

(a) (Optional) First write the electrostatic Maxwell equations $\nabla \cdot \boldsymbol{E} = \rho$ and $\nabla \times \boldsymbol{E} = 0$ using tensor notation, and explain why $\partial_i E_j = \partial_j E_i$.

(b) Within the limits of electrostatics, show that the electric force on a charged body is related to a surface integral of the (electric) stress tensor:

$$F^j = \int_V d^3\boldsymbol{r}\, \rho(\boldsymbol{r})\, E^j = -\int_S dS\, n_i T_E^{ij} \tag{6}$$

where $T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$, i.e. show that $\rho E^j = -\partial_i T_E^{ij}$

The Electric Stress Tensor

a) Since $\nabla \times E = 0$ we have

$$\varepsilon^{ijk} \partial_j E_k = 0$$

multiplying by ε_{ilm}, using

$$\varepsilon_{ilm} \varepsilon^{ijk} = \delta^j_l \delta^k_m - \delta^k_l \delta^j_m$$

gives

$$\partial_l E_m - \partial_m E_l = 0$$

b) The force per volume is

$$f^j = \rho E^j$$

$$= (\partial_l E^l) E^j$$

$$= \partial_l (E^l E^j) - E^l \partial_l E^j$$

use part a

$$= \partial_l (E^l E^j) - E^l \partial^j E_l$$

$$= \partial_l (E^l E^j) - \partial^j \left(\frac{E^2}{2}\right)$$

$$f^j = -\partial_l \left(-E^l E^j + \frac{1}{2} \delta^{lj} E^2\right)$$

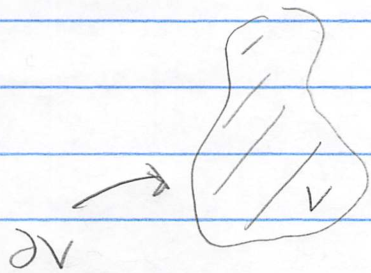
So

$$f^j = -\partial_i T^{ij} \quad \text{where} \quad T^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

So

$$F^j = \int_V \rho E^j = \int_V -\partial_i T^{ij}$$

$$= -\int_{\partial V} da_i \, T_E^{ij}$$



Problem 3. A stress tensor tutorial

Do not turn in the optional parts.

(a) (Optional) Consider a plane of charge with surface charge density σ, use the boundary conditions (i.e. Gauss Law) to show that the electric field on either side is $\sigma/2$

(b) (Optional) Consider an ideal infinite parallel plate capacitor with surface charge densities σ and $-\sigma$ respectively. Without using the stress tensor machinery, show that the force per area on each of the plates is $\sigma^2/2$

(c) (Optional) Consider a charged perfectly conducting solid object of any shape. Explain physically why the electric field is: (i) normal to the surface, (ii) zero on the inside, (iii) and equal to

$$\boldsymbol{E} = \sigma\, \boldsymbol{n} \qquad \text{or} \qquad E^i = \sigma\, n^i \tag{7}$$

(d) Without using the stress tensor machinery, show that the force per area on the walls of any metal surface is $\sigma^2/2$. (*Hint:* how large is the self field? Use part (a).)

The physics of the stress tensor is easy illustrated by knowing that the stress tensor of ideal gas is $T^{ij}_{\rm gas} = p\,\delta^{ij}$, where p is the pressure (force per area). Thus, if one considers a wall separating two gasses of left and right pressures p_L and p_R (*i.e.* the normal vector is[1], $n^j = \delta^{jx}$), then the net force per area on the wall is

$$n_i T^{ij}_L - n_i T^{ij}_R = (p_L - p_R)\, n^j \tag{8}$$

Note: that it is only the differences in the stress tensor which are physically important.

(e) (Optional) Recall that the net force on any object

$$F^j = -\oint dS\, n_i T^{ij}\,, \tag{9}$$

which we derived from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0\,. \tag{10}$$

Deduce from this that the net force per area on a wall separating two regions is

$$n_i (T^{ij}_{\rm out} - T^{ij}_{\rm in})\,. \tag{11}$$

(f) Using the electric stress tensor $T^{ij}_E = -E^i E^j + \frac{1}{2}E^2 \delta^{ij}$, show that the force per area on the surface of a charged metal object is

$$\text{force-per-area} = \frac{\sigma^2}{2} n^j \tag{12}$$

where $\boldsymbol{n}$ points from inside the metal to out.

[1] The notation is to confuse/educate you – I could have written $\boldsymbol{n} = (1, 0, 0)$ or $\boldsymbol{n} = \hat{\boldsymbol{x}}$.

(g) Now consider a charged and isolated parallel plate capacitor with charge per area $-\sigma$ and $+\sigma$ on the left and right plates (so that the normal is $n^j = \delta^{jx}$). A plane of charge with charge per area $\sigma/2$ lies halfway between the plates.

   (i) Compute all non-zero components of the stress tensor in the regions to the left and right of the plane of charge.

   (ii) Use the stress tensor to compute the force per area on the plane of charge, and show that it agrees with a simple minded approach.

Then

d) Take a portion of metal:

Fig 1L

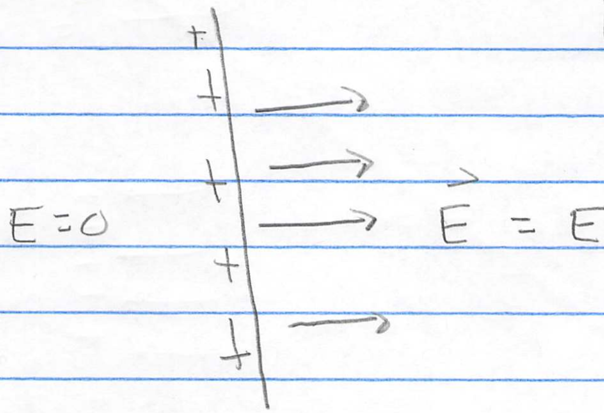
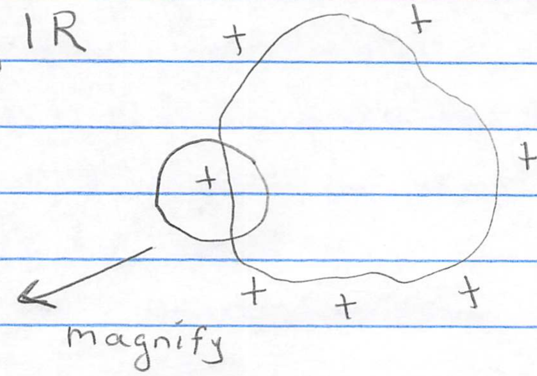


Fig 1R



The Electric field contains two contributions:
The field from the wall itsself (Fig 1L)

$$\vec{E}_{self} = \frac{\sigma}{2} \vec{n}$$

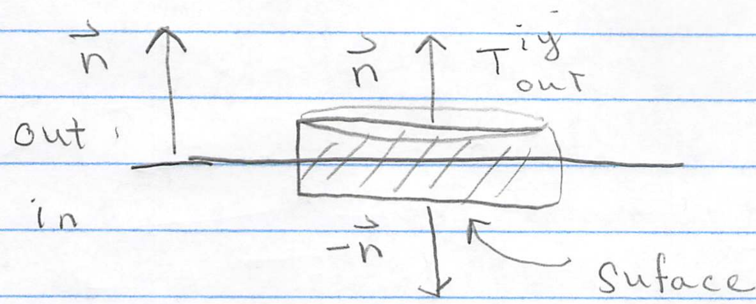
And the field from the remaining charges (Fig 1R). Since $\vec{E} = \sigma \vec{n}$. The field of the remaing charges is

$$E_{other} = \frac{\sigma}{2} \vec{n}$$

So the force per area is

$$\text{Force per area} = \sigma \vec{E}_{other} = \frac{\sigma^2}{2} \vec{n}$$

e) Take a surface element:



surface area

normal

$dS\, n_i \equiv da_i$

From

$$F^j = -\oint dS\, n_i\, T^{ij}$$

We apply it to the gaussian pillbox

$$F^j = -A\, n_i (T^{ij}_{out}) - (A)(-n_i)\, T^{ij}_{in}$$

this is the normal on the bottom

Then

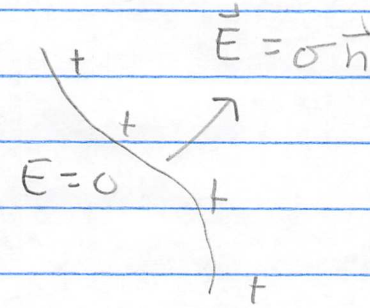
$$F^j = A\left(n_i T^{ij}_{in} - n_i T^{ij}_{out}\right)$$

$$\frac{F^j}{A} = n_i\left(T^{ij}_{in} - T^{ij}_{out}\right)$$

f)

f) We use the formulas:

$$E^i = \sigma n^i \quad , \quad n^i n_i = \vec{n} \cdot \vec{n} = 1$$



$$n_i(\cancelto{0}{T^{ij}_{in}} - T^{ij}_{out}) = \text{force/area}$$

So we have set $T^{ij}_{in} = 0$, since $\vec{E}_{in} = 0$

$$-n_i T^{ij}_{out} = \text{force per area}$$

$$-n_i\left(-E^i E^j + \frac{1}{2} E^2 \delta^{ij}\right) = \text{force per area}$$

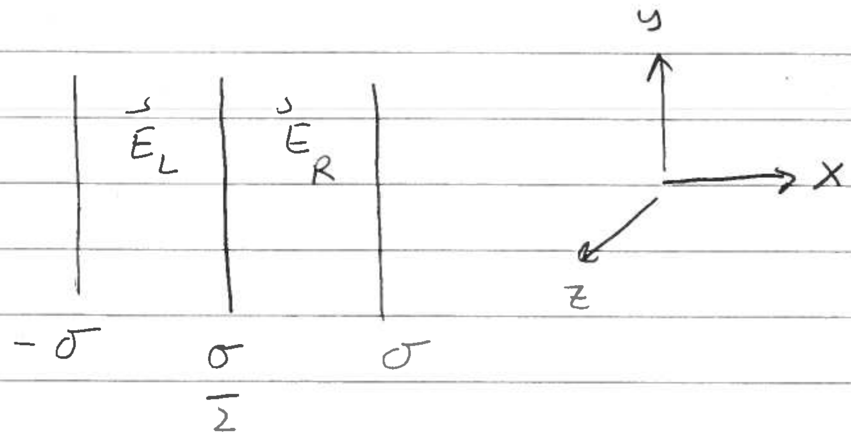
$$-n_i\left(-\sigma^2 n^i n^j + \frac{1}{2}\sigma^2 \delta^{ij}\right) = \text{''}$$

$$\sigma^2 n^j - \frac{1}{2}\sigma^2 n^j = \text{''}$$

$$\boxed{\frac{\sigma^2}{2} n^j = \text{force per area}}$$

Problem - Practice @ Stress Tensor

a)



$$\vec{E}_L = \underbrace{-\frac{\sigma}{2}\hat{x}}_{\text{Left plate}} \underbrace{-\frac{\sigma}{2}\hat{x}}_{\text{Right plate}} \underbrace{-\frac{\sigma}{4}\hat{x}}_{\sigma/2 \text{ plane}} = -\frac{5\sigma}{4}\hat{x} \equiv \left(\bar{E} - \frac{\Delta E}{2}\right)\hat{x}$$

$$\vec{E}_R = -\frac{\sigma}{2}\hat{x} - \frac{\sigma}{2}\hat{x} + \frac{\sigma}{4}\hat{x} = -\frac{3}{4}\sigma\hat{x} \equiv \left(\bar{E} + \frac{\Delta E}{2}\right)\hat{x}$$

So in the Left Region:

$$T^{ij} = -E^i E^j + \frac{1}{2}E^2 \delta^{ij}$$

Has non-zero components

$$T^{zz} = T^{yy} = \frac{1}{2}E^2 = \frac{25}{32}\sigma^2$$

$$T^{xx} = -\frac{25}{32}\sigma^2$$

The right side has non-zero components

$$T^{zz} = T^{yy} = \frac{9}{32}\sigma^2$$

$$T^{xx} = -\frac{9}{32}\sigma^2$$

So:

$$\frac{F^x}{A} = -\left(T^{xx}_R - T^{xx}_L\right)$$

$$= -\left(\frac{-9}{32} + \frac{25}{32}\right)\sigma^2 = -\frac{\sigma^2}{2}$$

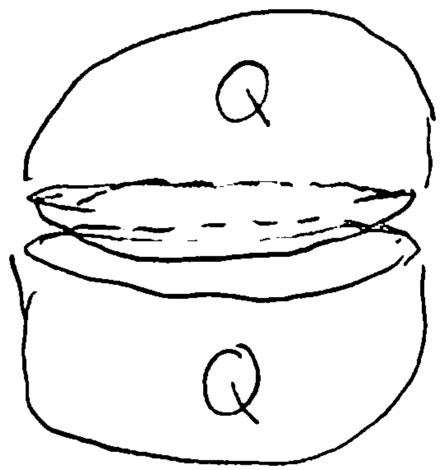
This is as expected the field produced by $-\sigma + \sigma$, is $\sigma = E_{plates}$ (in $-\hat{x}$ direction) and the charge is $\sigma/2$, so expect force

$$\frac{F^x}{A} = \underbrace{(-\sigma)}_{E_{plates}}\underbrace{\left(\frac{\sigma}{2}\right)}_{\text{charge/area}} = -\frac{\sigma^2}{2}$$

Problem 4. Practice with the stress tensor

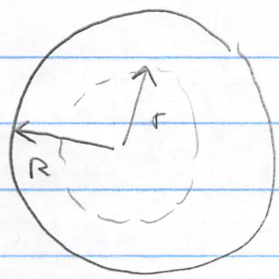
(a) Calculate the force between two (solid and insulating) uniformly charged hemispheres each with total charge Q and radius R that are separated by a small gap as shown below. You should find

$$F = \frac{3Q^2}{16\pi R^2} \tag{13}$$



Forces on a Sphere

First we find the E field inside the sphere



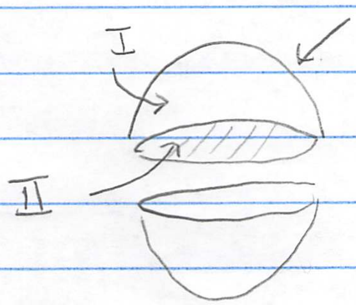
Using Gauss Law with the dashed surface:

$$\int \vec{E} \cdot d\vec{a} = Q_{in}(r)$$

$$E \, 4\pi r^2 = 2Q \, \frac{r^3}{R^3}$$

$$E = \frac{2Q}{4\pi R^2} \left(\frac{r}{R} \right)$$

Now we have to integrate $-\int dS \, n_i \, T^{ij}$ over the surface:



integrate over the surface with I and II.

The net force is in the z-direction, so we will set $j = z$

For surface I, $\vec{n} = \hat{r}$, and on the surface

$$\vec{E} = \frac{Q}{4\pi R^2}\,\vec{n}.$$

So, defining $E_0 \equiv 2Q/4\pi R^2$, we have:

$$-n_i T^{ij} = -n_i\left(-E_0^2 n^i n^j + \frac{1}{2}E_0^2 \delta^{ij}\right)$$

$$-n_i T^{ij} = \frac{1}{2}E_0^2\, n^j$$

We are interested in the z-component of the force;

$$-n_i T^{iz} = \frac{1}{2}E_0^2\, n^z = \frac{1}{2}E_0^2 \cos\theta$$

We have used

$$\vec{n} = \hat{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

Integrating

$$F^z_{I} = \int_{\substack{\text{hemi}\\ \text{sphere}}} \overbrace{R^2\, d(\cos\theta)\, d\phi}^{dS}\ \overbrace{\left(\frac{1}{2}E_0^2\cos\theta\right)}^{-n_i T^{iz}}$$

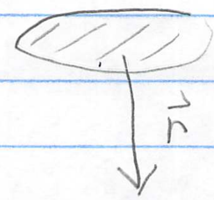
$$F^z_{I} = \frac{Q^2}{8\pi R^2}$$

Then we add the contribution from the bottom:

On the bottom,

$\vec{n} = -\hat{z}$, i.e. the normal is down

$= (0,0,-1)$



$\vec{n}$

So

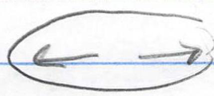
$$-n_i T^{iz} = +T^{zz}$$

and,

$$T^{zz} = -E^z E^z + \frac{1}{2} E^2 \delta^{zz}$$

$$= 0 + \frac{1}{2}(E_o)^2 \left(\frac{r}{R}\right)^2$$

we used the fact that $E^z = 0$ (on surface) and the magnitude of $\vec{E}$ is



$$|\vec{E}| = E_o \left(\frac{r}{R}\right) \qquad E_o \equiv \frac{2Q}{4\pi R^2}$$

So

$$F^z_{\mathrm{II}} = \int_0^R 2\pi r\, dr\, T^{zz}$$

$$= 2\pi \frac{1}{2} E_o^2 \int_0^R r\, dr \left(\frac{r}{R}\right)^2$$

$$F^z_{\mathrm{II}} = Q^2 / 16\pi R^2$$

Summarizing

$$F^z = F^z_{\mathrm{I}} + F^z_{\mathrm{II}} = \frac{Q^2}{8\pi R^2} + \frac{Q^2}{16\pi R^2}$$

$$= \frac{3Q^2}{16\pi R^2}$$

Problem 5. Green function of a sphere

Consider a grounded, metallic, hollow spherical shell of radius R. A point charge of charge q is placed at a distance, a, from the center of the sphere along the z-axis. For simplicity take $a > R$.

(a) Start by momentarily setting $R = 1$, and therefore measure all lengths in units of R. a is then shorthand for a/R in this system of units. With these units, show that the distance from the point $\boldsymbol{r} = a\,\hat{\mathbf{z}}$ to any point, $\boldsymbol{n}$, on the surface of the sphere is equal (up to a constant factor of a) to the distance from a point at $\boldsymbol{r} = (1/a)\,\hat{\mathbf{z}}$ to the same point $\boldsymbol{n}$ on the sphere.

(b) Use the result of part (a) to construct the Green function of the grounded sphere of radius R using images, *i.e.* find the potential due to a point charge at $\boldsymbol{r} = a\hat{\mathbf{z}}$ in the presence of a grounded sphere.

(c) Compute the surface charge density, and show that it is correct by directly integrating to find the total induced charge on the sphere of part (b). You should find that the total induced charge is equal to the enclosed image charge (why?). Please do not use Mathematica to do integrals.

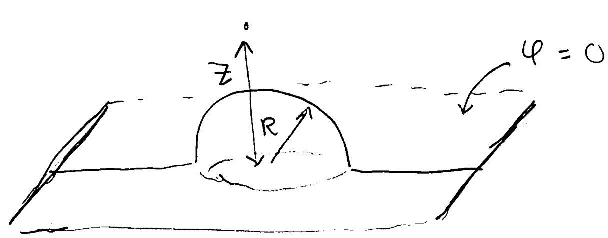
(d) Now consider a point charge of charge q at a point $\boldsymbol{r} = z\hat{\mathbf{z}}$ above a metallic hemisphere of radius R in contact with a grounded plane (see below). Determine the force on the charge as a function of z. You should find that at a distance $z = 2R$ the force is

$$F^z = -\frac{Q^2}{4\pi R^2}\left(\frac{737}{3600}\right) \tag{14}$$

(e) Show that at large distances, z, the Taylor series expansion for F^z is

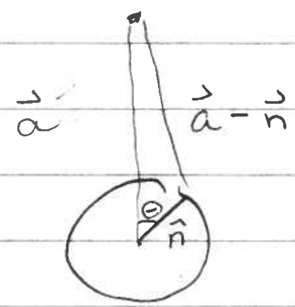
$$F^z \simeq \frac{Q^2}{4\pi R^2}\left[\frac{-1}{4u^2} - \frac{4}{u^5} + \ldots\right]$$

where $u = z/R$. Explicitly explain the coefficients of the series expansion (*i.e.* the $-1/4$ and -4) in terms of the multiple moments of the image solution.



Green - fcn on a sphere

a)



$$d(a,\theta) = |\vec{a} - \hat{n}|$$

$$d(a,\theta) = (a^2 + 1 - 2a\cos\theta)^{1/2}$$

$$= a\left(1 + \frac{1}{a^2} - \frac{2}{a}\cos\theta\right)^{1/2}$$

$$= a\, d\left(\frac{1}{a}, \theta\right)$$

b) Then the Green fcn of a grounded sphere of radius 1 is

$$\varphi = \frac{q}{4\pi|\vec{r} - a\hat{z}|} - \frac{q/a}{4\pi|\vec{r} - \frac{1}{a}\hat{z}|}$$

So that if $\vec{r} = \vec{n}$ then $\varphi = 0$. For $R \neq 1$ we restore units

$$\varphi \longrightarrow \frac{\varphi}{R}$$

$$\vec{r} \longrightarrow \frac{\vec{r}}{R}$$

$$\vec{a} \longrightarrow \frac{\vec{a}}{R}$$

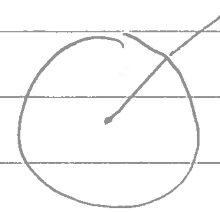
and find

$$\varphi = \frac{q}{4\pi R \left| \frac{\vec{r}}{R} - \frac{a\hat{z}}{R} \right|} - \frac{q\,R/a}{4\pi R \left| \frac{\vec{r}}{R} - \frac{R\hat{z}}{a} \right|}$$

$$\boxed{\varphi = \frac{q}{4\pi |\vec{r} - a\hat{z}|} - \frac{q\,R/a}{4\pi |\vec{r} - R^2/a\,\hat{z}|}}$$

c) Computing the surface charge density

$$\sigma = E_r$$



$$\sigma = -\frac{\partial \varphi}{\partial r}\bigg|_{r=R}$$

$$\varphi = \frac{q}{4\pi (r^2 + a^2 - 2ar\cos\theta)^{1/2}} - \frac{q\,R/a}{4\pi \left(r^2 + (R^2/a)^2 - 2r\frac{R^2}{a}\cos\theta\right)^{1/2}}$$

$$-\frac{\partial \varphi}{\partial r} = \frac{q\,(r - a\cos\theta)}{4\pi (r^2 + a^2 - 2ar\cos\theta)^{3/2}}$$

$$\left. - \frac{q\,R/a\,(r - R^2/a\cos\theta)}{4\pi (r^2 + (R^2/a)^2 - 2r R^2/a\cos\theta)^{3/2}} \right|_{r=R}$$

or

$$\sigma = \frac{q\,(R - a\cos\theta)}{4\pi\, d(a,\theta)^3} - \frac{q\, R/a\,(R - 2R^2/a\cos\theta)}{4\pi\, d\left(\frac{R^2}{a},\theta\right)^3}$$

Using:

$$d(a,\theta) = \frac{a}{R}\, d\left(\frac{R^2}{a},\theta\right)$$

$$\sigma = \frac{q\,(R - a\cos\theta)}{4\pi d^3} - \frac{q\,(R/a)\,(R - R^2/a\cos\theta)}{4\pi \left(\frac{R}{a}\right)^3 d^3}$$

$$\sigma = \frac{q}{4\pi d^3}\left[R\left(1 - \frac{a}{R}\cos\theta\right) - \left(\frac{a}{R}\right)^2 R\,(1 - R/a\cos\theta)\right]$$

$$\boxed{\sigma = \frac{q}{4\pi d^3} R\left[1 - \left(\frac{a}{R}\right)^2\right]}$$

Then integrating

$$q_{ind} = \int_{\text{sphere}} \sigma\; R^2\, d\Omega$$

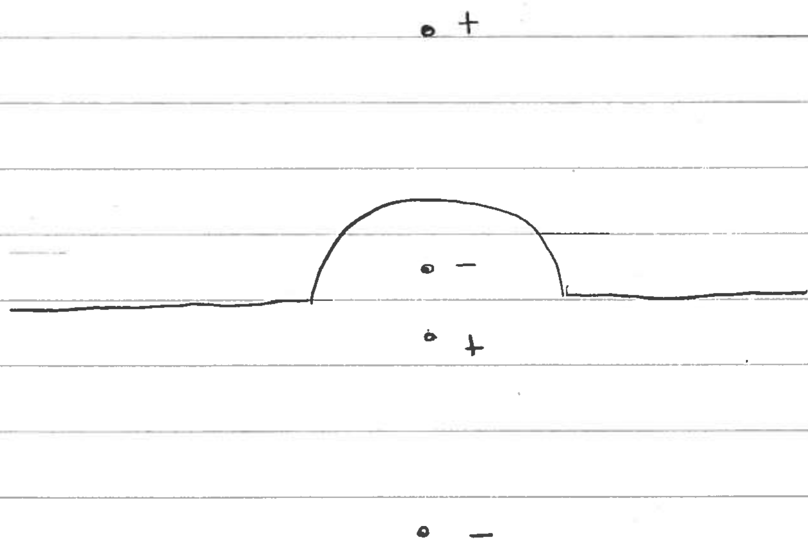
$$= 2\pi R^2 \int_{-1}^{1} dx \; \frac{qR\,(1 - a^2/R^2)}{4\pi\,(R^2 + a^2 - 2aRx)^{3/2}}$$

$$= \frac{2\pi\, q R^3 (1 - a^2/R^2)}{4\pi\, aR}\left[-(R^2 + a^2 - 2aRx)^{-1/2}\right]_{-1}^{1}$$

$$q_{ind} = \frac{q}{2}\frac{1}{a}(R^2 - a^2)\left[\frac{-1}{(a-R)} + \frac{1}{(a+R)}\right]$$

$$\boxed{q_{ind} = -q\frac{R}{a}}$$

d)



So with $u = z/R$

$$F^x = \frac{Q^2}{4\pi R^2}\left[\frac{-1/u}{(u-1/u)^2} + \frac{1/u}{(u+1/u)^2} - \frac{1}{(2u)^2}\right]$$

So with $u = 2$

$$F^x = \frac{Q^2}{4\pi R^2}\left(-\frac{737}{3600}\right)$$

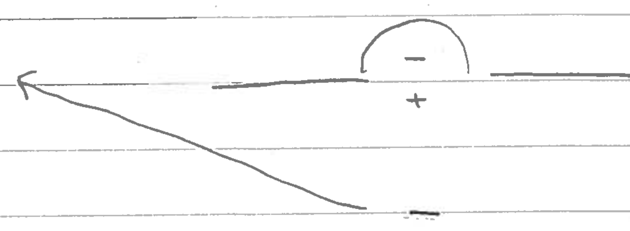
Then expanding for $u \to \infty$

$$F^x = \left(-\frac{1}{4u^2} - \frac{4}{u^5} \right) \times \frac{Q^2}{4\pi R^2}$$

The interpretation is the following:

(1) The image charges have a monopole moment

$$Q_{imag} = -Q$$



This gives a field $E = \frac{-Q}{4\pi r^2}$ and $F = -\frac{Q^2}{4\pi (2z)^2}$

(2) The two charges in the center form a dipole

$$E_{dip} = \frac{3(n \cdot p) - p}{4\pi r^3} \Rightarrow \frac{2p}{4\pi r^3} \quad \text{on axis}$$

The sign of the dipole is negative and has magnitude:

distance $= 2\frac{R^2}{z}$ $\longrightarrow$ $\downarrow$ $-\ QR/z$, $+\ QR/z$

$$p = -2Q(R/z)\frac{R^2}{z}$$

So the electric field from the dipole

$$E_{dip} = - \frac{4QR^3}{z^2} \cdot \frac{1}{4\pi z^3}$$

So the force due to the dipole is:

$$F_{dip} = - \frac{4Q^2}{4\pi R^2} \left(\frac{R}{z}\right)^5$$

So

$$F^{x} = + \frac{Q^2}{4\pi R^2} \left[\frac{-R^2}{4z^2} - 4\left(\frac{R}{z}\right)^5 + \ldots \right]$$