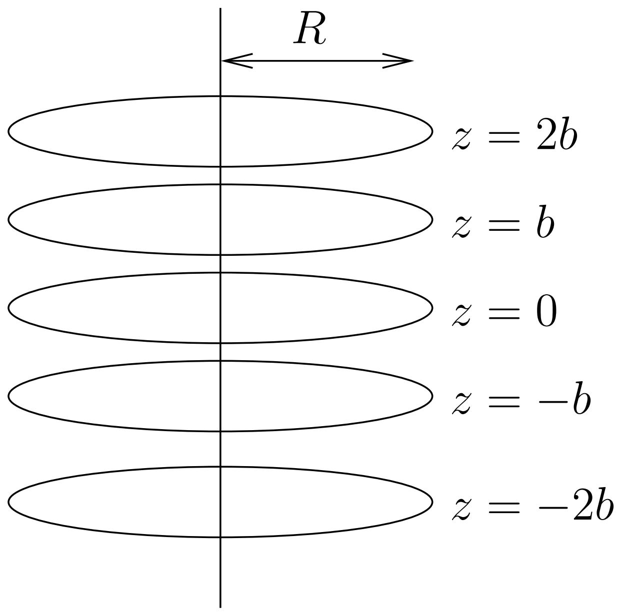
Problem 1. A Periodic Array of Charged Rings

Consider a periodic array of charged rings of radius R and separation b, so that the z-coordinates of the rings are $z = 0, \pm b, \pm 2b, \ldots$. Each ring has charge Q. We will find the potential below



(a) First classify the homogeneous solutions to the Laplace equation in cylindrical coordinates with azimuthal symmetry using separation of variables. Show that a typical homogeneous solution can be written $\varphi(z, \rho) = R(\rho)Z(z)$ and determine the equations that $R(\rho)$ and $Z(z)$ satisfy. You should find

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + k^2 R\right] = 0 \tag{1}$$

(i) What are the solutions to the $Z(z)$ equation? For z-periodic functions with period b what are the allowed values of k

(ii) Where are the singular points of radial differential equation, Eq. (1)?

(iii) (Do not hand this in) The two solutions to Eq. (1) are the modified Bessel functions $I_0(k\rho)$ and $K_0(k\rho)$. Look up these functions and make a graph of them. Record (from the internet or Mathematica) a series expansion at $x = 0$ and $x = \infty$ for these functions. Note the following:

i. At $x = 0$, one function is regular and one function is irregular.

ii. At $x = \infty$ the two solutions exchange roles, with the regular function at $x = 0$ becoming irregular at $x = \infty$ and the regular function at $x = \infty$ becoming irregular at $x = 0$.

Why is this the expected behavior?

(iv) For the differential equation

$$\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y(x)=0 \tag{2}$$

where $p(x)$ is positive definite, show that the $p(x)W(x)$ is constant, where $W(x)$ is the Wronskian of the two solutions to the differential equation. Determine $I_0'(x)K(x)-K'(x)I_0(x)$ up to a constant. Determine the (conventional) constant by using the series expansion for the modified Bessel functions at $x=0$.

To summarize we have shown that for azimuthally symmetric functions the solutions to the homogeneous Laplace equation take the form

$$\varphi(z,\rho)=\sum_n\left(A_nI_0(k_nx)+B_nK_0(k_nx)\right)e^{ik_nx}\,, \tag{3}$$

where for periodic functions only discrete values of k are allowed. At this point you should basically understand the course notes – Appendix D.3, which treated the case where the potential vanished at $z=0$ and $z=L$, and also allowed for non-azimuthally symmetric potentials. Now return to the problem at hand – the periodic array of rings.

(b) This problem is solved by exploiting the periodic nature of the problem, writing the charge density and the potential as a Fourier series. Use completeness to show that that the charge density is

$$\rho(\boldsymbol{x})=\frac{Q}{2\pi R}\delta(\rho-R)\frac{1}{b}\sum_{n=-\infty}^{\infty}e^{ik_nz} \tag{4}$$

where $k_n=2\pi n/b$.

(c) Solve for the potential inside and outside the rings, and use the jump condition to relate the two solutions. Show that the potential outside of the rings is

$$\varphi(\boldsymbol{x})=\frac{Q}{2\pi b}\left[-\ln\rho+2\sum_{n=1}^{\infty}\cos(k_nz)I_0(k_nR)K_0(k_n\rho)\right] \tag{5}$$

(d) For ρ large show that

$$\varphi(\boldsymbol{x})\simeq\frac{Q}{2\pi b}\left[-\ln\rho+\sqrt{\frac{b}{\rho}}\cos(\frac{2\pi z}{b})I_o(2\pi R/b)e^{-2\pi\rho/b}\right] \tag{6}$$

and explicitly interpret the leading term, $-\ln\rho$, and its coefficient, $Q/(2\pi b)$. Qualitatively explain the behaviour of the subleading term for large and small R/b.

Problem 2. A dielectric cylinder in an external field with a gradient

An infinitely long dielectric cylinder of radius a (centered at the origin with axis along the z axis) is placed in an approximately constant external electric field in the x direction. The external electric field contains a constant small gradient in the x direction, $\partial_x E_x \equiv E_o'$. The external potential is described by

$$\varphi_{\rm ext}(\boldsymbol{r}) = -E_o x - \frac{1}{2}E_o'\left(x^2 - y^2\right) \tag{7}$$

The gradient is small since $E_o' a \ll E_o$.

(a) (**Optional**) Separate variables in cylindrical coordinates with $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Show that the general solution to the Laplace equation takes the form

$$\varphi = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty}\left(A_n\rho^n + \frac{B_n}{\rho^n}\right)\cos(n\phi) + \sum_{m=1}^{\infty}\left(C_m\rho^m + \frac{D_m}{\rho^m}\right)\sin(m\phi) \tag{8}$$

When I first started writing this problem, I set $\varphi_{\rm ext}(\boldsymbol{r}) = -E_o x - \frac{1}{2}E_o' x^2$, what is wrong with this?

(b) Determine the potential both inside and outside the cylinder including the first correction due to the field gradient, E_o'.

(c) Determine the surface charge induced on the cylinder including the first correction due to the field gradient.

(d) Use the stress tensor to calculate the net force on the cylinder to first order in the gradient. You can either do this problem using Mathematica, or by pure thinking. The pure thinking method consists of the following. Write the electric field as a sum of two pieces

$$E = E_1 + E_2 \tag{9}$$

The stress tensor formed from these fields is of the form:

$$T_{\rm tot} = T_{11} + 2T_{12} + T_{22} \tag{10}$$

where, for example, T_{11} is the stress tensor from E_1. Argue physically that only T_{12} contributes, and compute this contribution.

Problem 3. A point charge and a semi-infinite dielectric slab

A point charge of charge q in vacuum is at the origin $\boldsymbol{r}_o = (0,0,0)$. It is separated from a semi-infinite dielectric slab filling the space $z > a$ with dielectric constant $\epsilon > 1$. When evaluating the potential for $z < a$, an image charge solution is found by placing an image charge at $z = 2a$. When evaluating the potential for $z > a$ we place an image charge at the origin. The full image solution is

$$\varphi(\boldsymbol{r}) = \begin{cases} \frac{q}{4\pi|\boldsymbol{r}|} - \frac{\beta q}{4\pi|\boldsymbol{r}-2a\hat{\boldsymbol{z}}|} & z < a \\ \frac{\beta' q}{4\pi\epsilon|\boldsymbol{r}|} & z > a \end{cases} \tag{11}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and $\beta' = (2\epsilon)/(1+\epsilon)$

(a) Sketch a picture of the resulting electric field lines.

(b) Quite generally show that the electric field lines refract at a discontinuous interface

$$\frac{\tan\theta_I}{\epsilon_{\mathrm{I}}} = \frac{\tan\theta_{\mathrm{II}}}{\epsilon_{\mathrm{II}}} \tag{12}$$

where θ_{I} and θ_{II} are the angles between the normal pointing from I to II and the electric fields in region I and region II, and ϵ_{I} and ϵ_{II} are the dielectric constants.

Problem 4. A Dielectric slab intervenes.

This problem will calculate the force between a point charge q in vacuum and a dielectric slab with dielectric constant $\epsilon > 1$. The point charge is at the origin $\boldsymbol{r}_o = (x_o, y_o, z_o) = (0,0,0)$, but we will keep x_o, y_o, z_o for clarity. The slab lies between $z = a$ and $z = a + \delta$ with $a > 0$ and has infinite extent in the x, y directions

(a) Write the free space Green function as a Fourier transform

$$\frac{q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \cdot (\boldsymbol{r}_\perp - \boldsymbol{r}_{o\perp})} g^o_{\boldsymbol{k}_\perp}(z, z_o) \tag{13}$$

and show that the free space green function in fourier space is

$$g^o_{\boldsymbol{k}_\perp}(z, z_o) = \frac{e^{-k_\perp |z - z_o|}}{2k_\perp} \tag{14}$$

(b) Now consider the dielectric slab and write the potential produced by the point charge at $z_o = 0$ as a Fourier transform

$$\varphi(\boldsymbol{r}_\perp, z) = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \boldsymbol{r}_\perp} g_{\boldsymbol{k}_\perp}(z)\,, \tag{15}$$

and determine for $g_{k_\perp}(z)$ by solving in each region, matching across the interfaces, and by analyzing the jump at z_o. Show that for $z < 0$ and $0 < z < a$

$$g_{\boldsymbol{k}_\perp}(z) = \begin{cases} \frac{e^{kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & z < 0 \\ \frac{e^{-kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & 0 < z < a \end{cases} \tag{16}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and we have written $k = k_\perp$ to lighten the notation.

(c) Checks:

(i) Show that for $\delta \to \infty$ the potential for $z < a$ is in agreement with the results of the previous problem.

(ii) Show that when $\epsilon \to \infty$ (when the dielectric becomes almost metallic) you get the right potential.

(d) Show that the electric potential for region $z < a$ can be written

$$\varphi = \varphi_{\text{ind}} + \frac{q}{4\pi r} \tag{17}$$

where φ_{ind} is the induced potential and is regular at $r = 0$. Show that the force on the point charge is

$$F^z = \beta \frac{q^2}{4\pi(2a)^2} \int_0^\infty du\, \frac{4u e^{-2u}(1 - e^{-2(\delta/a)u})}{1 - \beta^2 e^{-2(\delta/a)u}} \tag{18}$$

(e) Use a program such as mathematica to make a graph of the force $F^z/(\beta q^2/(4\pi(2a)^2))$ versus δ/a for $\beta = 0.1, 0.5, 0.9$ and sketch the result.