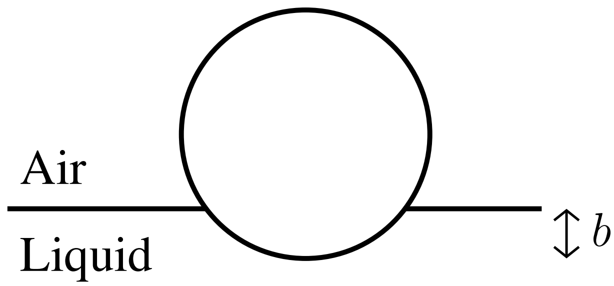
Problem 1. A half submerged metal sphere (UIC comprehensive exam)

A very light neutral hollow metal spherical shell of mass m and radius a is slightly submerged by a distance $b \ll a$ below the surface of a dielectric liquid. The liquid has mass density ρ and electrical permitivity ϵ. The liquid sits in air which has negligble density $\rho_o \ll \rho$, and the permitivity of air is approximately unity, $\epsilon_{\rm air} \simeq 1$. The pressure at the air liquid interface is p_0. Recall that stress tensor of an ideal fluid at rest is $T^{ij} = p(z)\delta^{ij}$ where $p(z)$ is the pressure as a function of z.



(a) Use the formalism of stress tensor to show that $p(z)$ increases as $p = p_0 + \rho g h$, where $h = -z$ is the depth below the surface, $z < 0$. Here p_0 is the pressure at the surface. Hint: what is the net force per volume for a static fluid?

(b) Use the formalism of stress tensor to prove that the boyancy force (for any shape) equals the difference in weight of the displaced fluid volume ΔV and the corresponding weight of the air:

$$F = (\rho - \rho_o) g \Delta V \simeq \rho g \Delta V.$$

Now a charge Q is added to the sphere, and the sphere becomes half submerged.

(c) Determine the potential, and the electrostatic fields E and D, in the top and lower halves of the sphere. Verify that all the appropriate boundary conditions are satisfied.

(d) What is the surface charge density on the top and lower halves of the sphere?

(e) Determine the electrostatic attractive force as a function of Q, a, and ϵ. What must Q be for the sphere to be half submerged? Make all reasonable approximations. Express your approximate result in terms of ρ, g, a, ϵ. Use dimensional reasoning to show that for a light sphere,

$$Q = \sqrt{\rho g a^5} \times \text{function of } \epsilon \,. \tag{1}$$

(f) (Optional) Estimate Q numerically for typical liquids.

Problem 2. A cylinder in a magnetic field (Jackson)

A very long hollow cylinder of inner radius a and outer radius b of permeability μ is placed in an initially uniform magnetic field $\boldsymbol{B}_o$ at right angles to the field.

(a) For a constant field B_o in the x direction show that $A^z = B_o y$ is the vector potential. This should give you an idea of a convenient set of coordinates to use.

Remark: See Wikipedia for a list of the vector Laplacian in all coordinates. Most often the vector Laplacian is used if the current is azimuthal and solutions may be looked for with $A_\phi \neq 0$ and $A_r = A_\theta = 0$ (or $A_\rho = A_z = 0$ in cylindrical coordinates). This could be used for example in Problem 3. Similarly if the current runs up and down, with $A_z \neq 0$ and $A_\rho = A_\phi = 0$, so that $\boldsymbol{B} = (B_x(x,y,z), B_y(x,y,z), 0)$ is independent of z, then the vector Laplacian in cylindrical coordinates $-\nabla^2 A_z$ is a good way to go.

(b) Show that the magnetic field in the cylinder is constant $\rho < a$ and determine its magnitude.

(c) Sketch $|\boldsymbol{B}|/|\boldsymbol{B}_o|$ at the center of the as function of μ for $a^2/b^2 = 0.9, 0.5, 0.1$ for $\mu > 1$.

Problem 3. Helmholtz coils (Jackson)

Consider a compact circular coil of radius a carrying current I, which lies in the $x-y$ plane with its center at the origin.

(a) By elementary means compute the magnetic field along the z axis.

(b) Show by direct analysis of the Maxwell equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = 0$ that slightly off axis near $z = 0$ the magnetic field takes the form

$$B_z \simeq \sigma_0 + \sigma_2 \left(z^2 - \tfrac{1}{2}\rho^2\right), \quad B_\rho \simeq -\sigma_2 z \rho \,, \tag{2}$$

where $\sigma_0 = (B_z^o)$ and $\sigma_2 = \frac{1}{2}\left(\frac{\partial^2 B_z^o}{\partial z^2}\right)$ are the field and its z derivatives evaluated at the origin. For later use give σ_0 and σ_2 explictly in terms of the current and the radius of the loop.

Remark: The magnetic field near the origin satisfies $\nabla \times \boldsymbol{B} = 0$, so $\nabla \cdot \boldsymbol{B} = 0$. We say it is harmonic function[1]. Because the function is harmonic, the taylor series of B on the z axis, is sufficient to determine the taylor series close to the z axis.

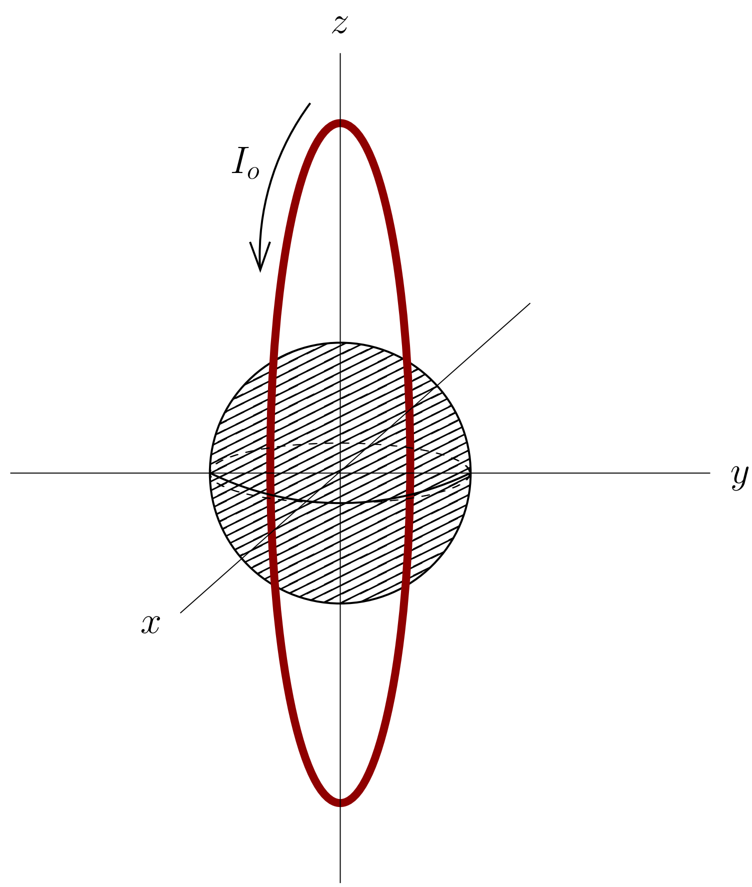
(c) Now consider a second identical coil (co-axial with the first), having the same magnitude and direction of the current, at a height b above the first coil, where a is the radii of the rings. With the coordinate origin relocated at the point midway between the two centers of the coils, determine the magnetic field on the z-axis near the origin as an expansion in powers of z to z^4. Use mathematica if you like. You should find that the coefficient of z^2 vanishes when $b = a$

Remark For $b = a$ the coils are known as Helmholtz coils. For this choice of b the z^2 terms in part (c) are absent. (Also if the off-axis fields are computed along the lines of part (b), they also vanish.) The field near the origin is then constant to 0.1% for $z < 0.17\,a$.

[1] This means that $\boldsymbol{B}$ can be written $\boldsymbol{B} = -\nabla\psi$ where $-\nabla^2\psi = 0$

Problem 4. A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



(a) Determine the bound surface current on the surface of the sphere.

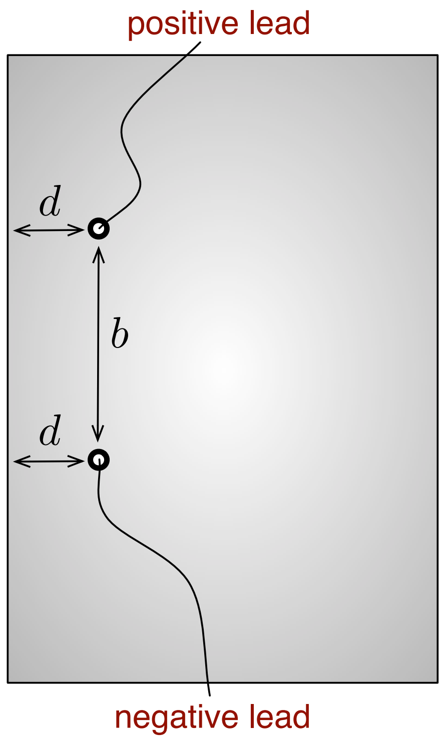
(b) Write down (no long derivations please) the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere by analogy with the spinning charged sphere disucssed in class.

(c) Show that your solution satisfies the boundary conditions of magnetostatics on the surface of the sphere.

(d) Compute the net-torque on the circular hoop. Indicate the direction and interpret.

Problem 5. Electrodes in an ohmic material filling half of space

Two small spherical electrodes of radius a are embedded in a semi-infinite medium of conductivity σ, each at a distance $d \gg a$ from the plane face of the medium and at a distance $b \gg a$ from each other.



(a) State the boundary conditions on all surfaces. Assume that the electrodes emit and absorb a total current I which is spread uniformly over the surface of the sphere. Do not assume that a is small for this part (but of course $a < b$ and $a < d$).

(b) Argue for small a the potential between the two electrodes satisfies

$$\nabla^2 \varphi = \frac{I}{\sigma} \delta^3(\boldsymbol{r} - \boldsymbol{r}_1) - \frac{I}{\sigma} \delta^3(\boldsymbol{r} - \boldsymbol{r}_2) \tag{3}$$

where $\boldsymbol{r}_1$ is the position of the emitting electrode while $\boldsymbol{r}_2$ is the position of the absorbing electrode.

(c) Find the resistance between the electrodes. Sketch the flow lines of current if the two electrodes are held at a potential difference ΔV. (Hint: use images to solve Eq. (3) with the right boundary conditions.)