Problem 1. Energy of a wire and rectangle (Jackson)

(a) Consider an infinitely long straight wire carying a current I in the z direction. Use the known mangetic field of this wire, and the integral form of $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

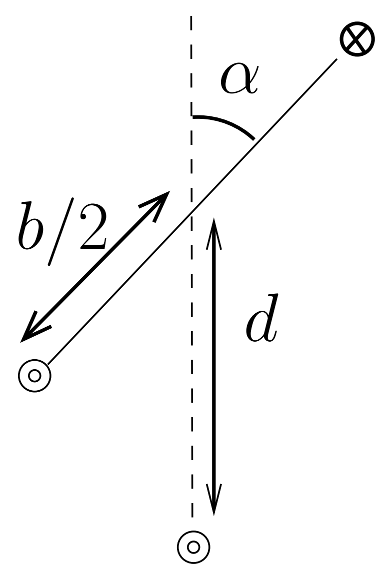
$$\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint \mathrm{d}\boldsymbol{\ell} \cdot \boldsymbol{A} \tag{1}$$

to show that the vector potential for an infinite current carrying wire in the Coulomb gauge is

$$A^z = \frac{-(I/c)}{2\pi} \log \rho + \text{const} \tag{2}$$

Check that the Coulomb gauge condition is satisfied.

(b) Now consider a flat right rectangular loop carrying a constant current I_1 that is placed near a long straight wire carrying a constant current I_2. The rectangular loop is oriented so that its center is a perpendicular distance d from the wire; the sides of length a are parrel to the wire and the sides of length b make an angle α with the plane containing the wire and the loops center (the dashed line below). In the schematic diagram below, the current I_2 in the long wire flows out of page. The orientation of I_1 is also indicated, i.e. the current lower edge of the rectangle (of length a) also comes out of the page.



Show that the interaction energy

$$W_{12} = \frac{I_1}{c} F_1 \tag{3}$$

(where F_1 is the magnetic flux from I_2 through the rectangular circuit carrying I_1), is

$$W_{12} = \frac{a I_1 I_2}{4\pi c^2} \ln \left[\frac{4d^2 + b^2 + 4db\cos\alpha}{4d^2 + b^2 - 4db\cos\alpha} \right] \tag{4}$$

(c) (Optional) Using energy considerations calculate the force between the loop and the wire for constant currents. (This requires some somewhat lengthy differentiation and is best done with mathematica) . You should find

$$F^x = \frac{I_1 I_2 a}{4\pi c^2}\left(\frac{8b(b^2-(2d)^2)\cos\alpha}{b^4+(2d)^4-2b^2(2d)^2\cos(2\alpha)}\right), \tag{5}$$

$$F^y = \frac{I_1 I_2 a}{4\pi c^2}\frac{1}{d}\left(\frac{4b(2d)(b^2+(2d)^2)\sin\alpha}{b^4+(2d)^4-2b^2(2d)^2\cos(2\alpha)}\right), \tag{6}$$

where in the figure above the x and y axes are respectively perpendicular and parallel to the separation d. For large $d \gg a, b$ the force expands to

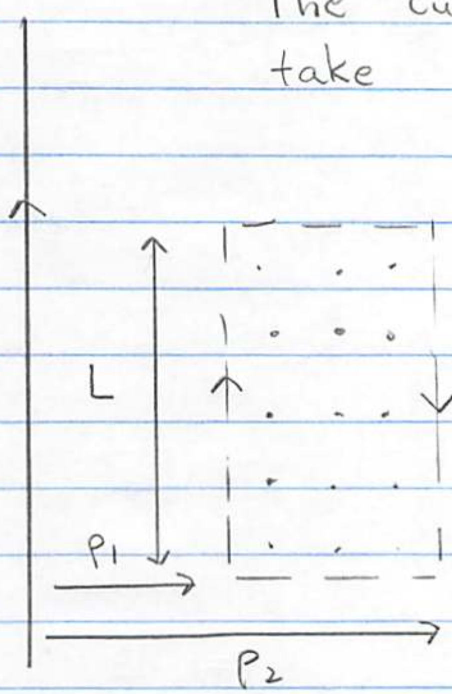
$$F^y = -\frac{I_1 I_2 ab\cos\alpha}{2\pi c^2 d^2} \tag{7a}$$

$$F^x = +\frac{I_1 I_2 ab\sin\alpha}{2\pi c^2 d^2} \tag{7b}$$

(d) (Optional) Check that for large distances $d \gg a, b$ the force computed in the previous sub-question agrees with the appropriate formula for a dipole in an external field.

(e) Show that when $d \gg a, b$ the interaction energy reduces to $W_{12} = \boldsymbol{m}\cdot\boldsymbol{B}$, where $\boldsymbol{m}$ is the magnetic moment of the loop. Explain the sign which is opposite from our previous result $U = -\boldsymbol{m}\cdot\boldsymbol{B}$. What is the difference between these two expressions for the energy of dipole in a magnetic field?

Energy of a wire



The current points in the z-direction, take $A^z(\rho)$. $B_\phi = \dfrac{I/c}{2\pi\rho}$

Integrating

$$\oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{a}$$

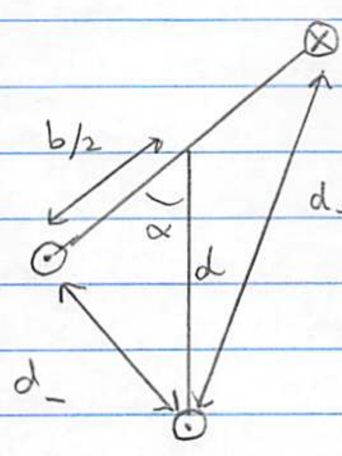
$$-A_2 L + A_1 L = L\int_{\rho_1}^{\rho_2} d\rho\, \frac{I/c}{2\pi\rho}$$

$$= \frac{LI/c}{2\pi}\left[\log\rho_2 - \log\rho_1\right]$$

Thus we compare and conclude

$$A^z = -\frac{I/c}{2\pi}\log\rho + \text{const}$$

b) Use the results of part(a).



Note from geometry:

$$d_\pm^2 = d^2 + (b/2)^2 \pm 2d\left(\frac{b}{2}\right)\cos\alpha$$

Then

$$W_{12} = \frac{I_1}{c} \oint d\vec{l}_1 \cdot \vec{A}_2$$

Using part (a) $A_2 = -\frac{(I_2/c)}{2\pi} \log \rho$

we find

$$W_{12} = -\frac{I_1 I_2 a}{2\pi c^2} \left[\log d_- - \log d_+ \right]$$ (see figure)

where $d_\pm$ are given by geometry

$$d_\pm = \left(d^2 + (b/2)^2 \pm 2d(b/2)\cos\alpha \right)^{1/2}$$

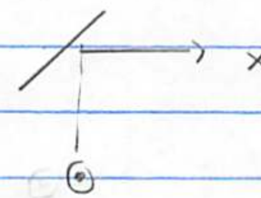
Thus we find

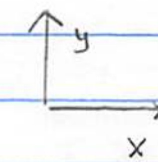
$$W_{12} = \frac{I_1 I_2 a}{4\pi c^2} \log \left(\frac{d^2 + (b/2)^2 + 2d(b/2)\cos\alpha}{d^2 + (b/2)^2 - 2d(b/2)\cos\alpha} \right)$$

Which is the answer quoted.

c) The force has two components

$$F^y = +\frac{\partial W_{12}}{\partial d}$$



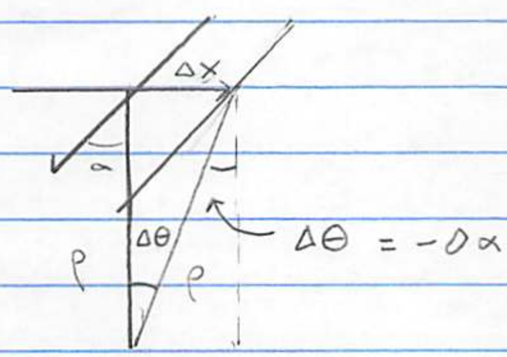


Straightforward algebra gives (Mathematica)

$$F^y = \left(\frac{8b\,(b^2 - (2d)^2)\cos\alpha}{b^4 + (2d)^4 - 2b^2(2d)^2\cos 2\alpha} \right) \frac{I_1 I_2 a}{4\pi c^2}$$

We can also compute the force in the x-direction. Under a small displacement in the x-direction we see that

$$\Delta x = \rho\,\Delta\theta = -\rho\,\Delta\alpha$$



Thus

$$+\frac{\partial W}{\partial x} = -\frac{1}{\rho}\frac{\partial W}{\partial \alpha}$$

Thus differentiating $F^x = -(\partial W/\partial\alpha)(1/d)$ (Mathematica)

$$F^x = \frac{1}{d}\left(\frac{4b\,(2d)\,(b^2 + (2d)^2)\sin\alpha}{b^4 + (2d)^4 - 2b^2(2d)^2\cos(2\alpha)} \right) \frac{I_1 I_2 a}{4\pi c^2}$$

d) Taking a taylor series $d \to \infty$

$$F^y \simeq -\left(\frac{2b\cos\alpha}{d^2}\right)\frac{I_1 I_2 a}{4\pi c^2}$$

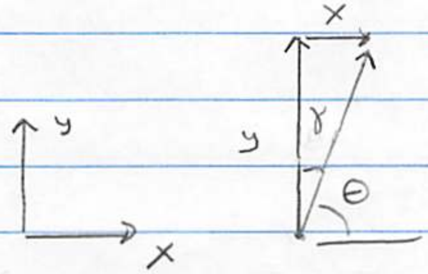
$$\boxed{F^y \simeq -\frac{I_1 I_2 ab}{2\pi c^2}\frac{\cos\alpha}{d^2} \qquad \text{Eq. 1}}$$

$$\boxed{F^x \simeq +\left(\frac{2b\sin\alpha}{d^2}\right)\frac{I_1 I_2 a}{4\pi c^2} \qquad \text{Eq. 2}}$$

To compare with the dipole form we use:

$$\vec{F} = (\vec{m}\cdot\nabla)\vec{B} \quad \text{or}$$

$$F^i = (m^l\cdot\partial_l)B^i = m^l\,\partial_i B^l$$



On axis $x = 0$ we have

$$B^x = -\frac{(I/c)}{2\pi y}$$

For x small:

$$\vec{B} = B_\rho\,\hat{\phi} = +\frac{I/c}{2\pi y}\left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)$$

So for small x (or $\theta \simeq \pi/2$ or γ small)

$$\vec{B} = B^x \hat{x} + B^y \hat{y} = \frac{(I/c)}{2\pi y} \left[-\hat{x} + \frac{x}{y} \hat{y} \right]$$

we used (see picture) that

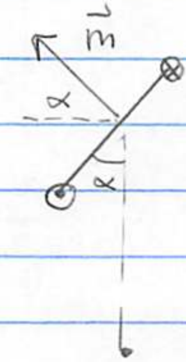
$$\cos\theta = \sin\gamma \simeq \frac{x}{y} \qquad \sin\theta \simeq \cos\gamma \simeq 1$$

Then we are ready to compute:

$$F^x = m^y \partial_y B^x$$

this vanishes as $x \to 0$

$$F^y = \underbrace{m^x \partial_x B^y}_{\text{this does not vanish as } x \to 0} + m^y \cancel{\partial_y B^y}^{\,0}$$



Then

$$F^x = m \sin\alpha \frac{\partial}{\partial y} \left(\frac{-I_2/c}{2\pi y} \right)$$

$$F^x = + \frac{m \sin\alpha \, I_2/c}{2\pi y^2}$$

This agrees with Eq. 2 on the previous page with $m = \frac{I_1}{c} ab$

Similarly with $m^x = -m\cos\alpha$

$$F^y = -m\cos\alpha \, \partial_x \left(\frac{I_2/c}{2\pi y} \, \frac{x}{y} \right)$$

$$F^y = -\frac{m\cos\alpha\,(I_2/c)}{2\pi y^2}$$

↑

This agrees with Eq. 1 on the two pages back

For the last part we take $d \to \infty$

$$W_{12} = \frac{I_1 I_2}{4\pi c^2} a \left\{ \log\left[4d^2 + b^2 + 4db\cos\alpha\right] - \log\left[4d^2 + b^2 - 4db\cos\alpha\right] \right\}$$

$$\log(4d^2 + b^2 + 4db\cos\alpha) \simeq \log(4d^2) + \log\left(1 + \frac{b}{d}\cos\alpha\right)$$

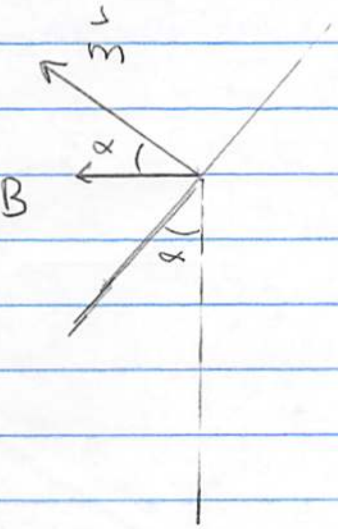
Yielding

$$\simeq \text{const} + \frac{b\cos\alpha}{d}$$

$$W_{12} = \frac{I_1 I_2}{4\pi c^2} a \frac{2b\cos\alpha}{d}$$

$$W_{12} = m\cos\alpha \frac{(I_2/c)}{2\pi d} = \vec{m}\cdot\vec{B} = mB\cos\alpha$$

A short picture shows that, indeed, the angle between $\vec{m}$ and $\vec{B}$ is α



The sign is positive because this (i.e. W_{12}) is the energy stored in the fields. The usual formula, is the energy required

$$U_{dip} = -\vec{m}\cdot\vec{B},$$

to bring a dipole with fixed currents from infinity to a specified location. U_{dip} ignores the work that must be done by the battery to maintain those currents in the face of the changing flux. W_{12} contains this work by the battery, U_{dip} does not.

- See Jackson

Problem 2. Dipole two ways

Consider two charges $\pm q$ moving a short distance on ℓ the z axis forming a dipole, i.e.

$$z_+(t) = \ell/2 \cos(\omega t) \tag{8}$$
$$z_-(t) = -\ell/2 \cos(\omega t) \tag{9}$$

This charged has dipole moment $p = q\ell \cos(\omega t)$ directed in the z direction

(a) Write down the electric field as a function of time.

(b) By treating the time dependent electric field as a "displacement current"

$$\frac{\boldsymbol{j}_D}{c} = \frac{1}{c}\partial_t \boldsymbol{E}, \tag{10}$$

determine the magnetic field using Ampere's Law

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \int_S d\boldsymbol{a} \cdot \frac{\boldsymbol{j}_D}{c} \tag{11}$$

with an appropriate Amperian loop. Assume that $\boldsymbol{B} = B(r,\theta)\,\hat{\phi}$ which can be justified from the symmetry of the problem. You should find

$$\boldsymbol{B} = \frac{\dot{\boldsymbol{p}} \times \boldsymbol{n}}{4\pi r^2 c} \tag{12}$$

(c) Consider the Biot Savat Law

$$\boldsymbol{B}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}(\boldsymbol{r}_0)}{c} \times \frac{(\boldsymbol{r} - \boldsymbol{r}_0)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_0|^3} \tag{13}$$

Show that if $\boldsymbol{j}$ is curl free $\nabla \times \boldsymbol{j} = 0$ the magnetic field is zero. Explain why the displacement current from an electrostatic field does not need to be included in the Biot-Savat Law.

Thus we see that the *electrostatic* displacement current is necessary for consistency of Ampere's Law but does not actually produce a magnetic field.

(d) Show (using a high school physics argument) that the current density integrated over small volume ΔV surounding the charges is

$$\boldsymbol{j}\Delta V = \partial_t \boldsymbol{p} \tag{14}$$

Use the Biot-Savat Law to determine the magnetic field. It shoud agree with (*b*).

(e) At what radius do the electric fields part (a) and the magnetic fields of part (b), (d) become equal in magnitude. At this radius the approximation scheme is no longer valid.

Problem 3. A rotating magnet

A magnetic dipole moment of magnitude m lying in the xy plane rotates about its center with angular velocity ω. It points in the x direction at time $t = 0$

(a) Find the electric and magnetic fields on the z axis as a function of time. Work to the lowest non-trivial order in inverse powers of c. (Hint: the easy way is to use the vector potential to determine $\boldsymbol{E}$)

(b) Estimate the magnitude of E/B at a given radius r. At what radius is the solution you found in part (a) valid? At what radius does it break down and why?

writing

$$\sin\theta \simeq \frac{\rho}{z} \qquad r \simeq z \qquad \text{we find}$$

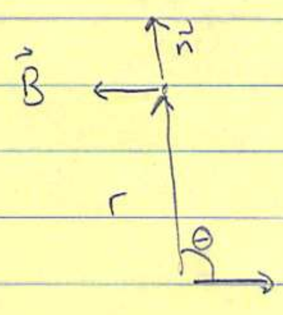
$$\boxed{\vec{E} = -\frac{\dot{\vec{m}}}{c}\,\frac{\rho}{4\pi z^3}\,\hat{\phi}} \quad \text{as before}$$

The solution starts here

b) Then

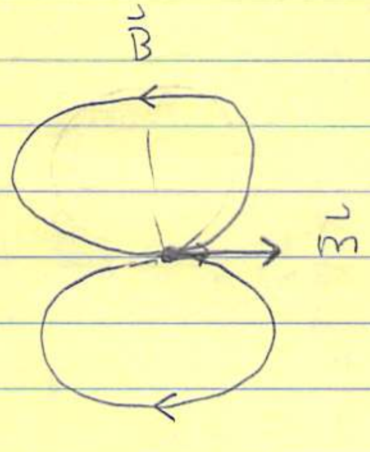
$$\vec{B}(t) = \frac{3\vec{n}\,(\vec{n}\cdot\vec{m}) - \vec{m}}{4\pi r^3}$$

but $\boldsymbol{m}$ points in the xy plane, while $\boldsymbol{n}$ is in the $\hat{\boldsymbol{z}}$, so direction



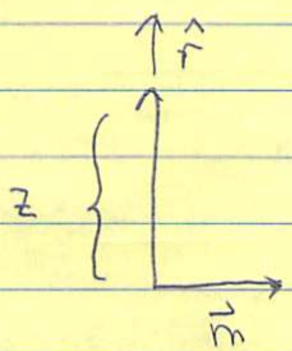
$$= \frac{-\vec{m}(t)}{4\pi r^3} = \boxed{\frac{-\vec{m}(t)}{4\pi z^3} = \vec{B}(t)}$$

Here $\vec{m}(t) = (\cos\omega t)\,\hat{x} + (\sin\omega t)\,\hat{y}$. Intuitively this magnetic field is just a dipole on its side



Now for the electric field

$$\vec{A} = \frac{\vec{m} \times \hat{r}}{4\pi r^2}$$



$$\hat{r} = \hat{z}$$

$$r = z$$

So

$$\vec{m}(t) \times \hat{z} = m\left[\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}\right] \times \hat{z}$$

$$= -m\cos\omega t\,\hat{y} + m\sin\omega t\,\hat{x}$$

So

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} = \frac{m}{4\pi z^2}\left(\frac{\omega}{c}\right)\left[\sin\omega t\,\hat{y} + \cos\omega t\,\hat{x}\right]$$

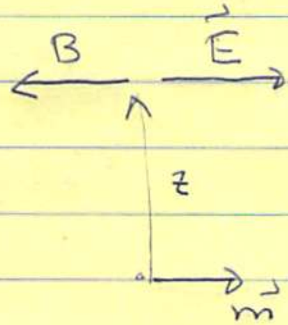
So

$$\boxed{\vec{E} = \frac{\vec{m}_{(t)}}{4\pi z^2}\left(\frac{\omega}{c}\right)}$$

$$ƛ = \frac{\text{wavelength}}{2\pi}$$

c) $\frac{E}{B} = \left(\frac{\omega}{c}\right) z$. Note $ƛ \equiv \frac{c}{\omega}$.

So for $z \sim \frac{1}{ƛ}$, E is comparable to $\vec{B}$, and quasistatics breaks down.



Problem 4. Electric and Magnetic fields of AC Solenoid

A cylindrical solenoid of high conductivity and radius a carries surface current $\mathbf{K} = K_o \cos(\omega t)\hat{\boldsymbol{\phi}}$

(a) Determine the electric and magnetic fields to the first non-vanishing order in the quasi-static approximation.

(b) Show that the magnetic field to the next-to-leading order in the quasi-static approximation outside the cylinder is

$$\Delta B = \delta B_z(\rho) - \delta B_z(\rho_{\max}) = \frac{K_o}{c}\cos(\omega t)\,\tfrac{1}{2}(\omega a/c)^2\left(-\log\frac{\rho}{a} + C\right) \tag{15}$$

where C $= \log \rho_{\max}/a$. Here we are quoting ΔB the difference between δB at ρ and δB at $\rho_{\max}$.

(c) The cutoff $\rho_{\max}$ arises because the quasi static approximation breaks down for large ρ where the physics of radiation becomes important. $\rho_{\max}$ should be of order $\rho_{\max} \sim c/\omega$. Explain qualitatively why the approximation breaks down for this radius.

Remark: Certainly $|\delta B_z(\rho_{\max})|$ is logarithmically smaller than $|\delta B_z(\rho)|$ for $\rho \sim a$. In a logarithmic approximation we can neglect $\delta B_z(\rho_{\max})$ and set $\rho_{\max} = c/\omega$ leading to

$$\delta B_z(\rho) \simeq \frac{K_o}{c}\cos(\omega t)\,\tfrac{1}{2}(\omega a/c)^2\left(-\log\frac{\rho}{a} + \log(c/(\omega a))\right) \tag{16}$$

Leading log accuracy may not be familiar to you. It just says that we are neglecting the constant inside the logarithm which is of order 1. Thus in this approximation,

$$\log(100/2) = \log(100) - \log(2) \simeq \log(100) \tag{17}$$

$$3.9 \simeq 4.6 \tag{18}$$

which is often good enough for government work. Bethe famously used such approximations to estimate the first QED corrections to the hydrodgen spectrum.

(d) Determine the magnetic field (to the next-to-leading order in the quasi-static approximation) inside the cylinder to logarithmic accuracy, and qualitatively sketch the complete magnetic field $B(\rho)/B_o$ where B_o is the leading order answer in the center of the cylinder.

Remark: Note that the ρ depence of part (b) and part (c) does not depend on the value of $C = \log(\rho_{\max}/a)$.

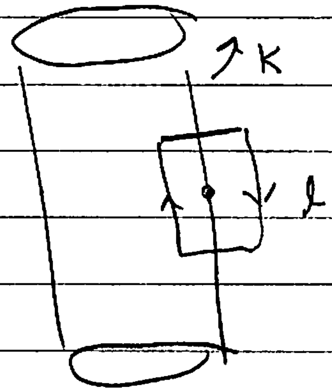
(e) Determine the vector and scalar potentials in the Coulomb and Lorentz gauges to the required order and accuracy to reproduce the electric and magnetic fields in part (a) and verify that you obtain the correct fields.

①

Magnetic Field from AC Solenoid

a) $\int \vec{B} \cdot d\vec{l} = I_{enc}/c$

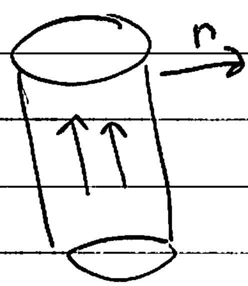
$$l(-\cancel{B^z_{out}}^{\,0} + B^z_{in}) = \frac{l}{c} K_o \cos\omega t$$



$$B^z_{in} = \frac{K_o}{c} \cos\omega t$$

Or from B.C.

$$\vec{n} \times (\cancel{\vec{B}_2}^{\,0} - \vec{B}_1) = \vec{K}$$



$\vec{n} \times (-\vec{B}_1) =$ in $\hat{\phi}$ direction

From:

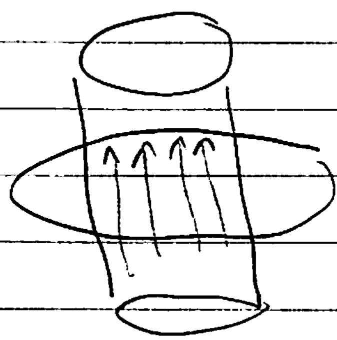
$$\int E \cdot dl = -\frac{1}{c} \int \partial_t \vec{B} \cdot da$$

$$E_\phi \, 2\pi\rho = -\frac{1}{c} \partial_t \left(\frac{K_o \cos\omega t}{c} \right) \pi\rho^2$$

$$E_\phi = \frac{K_o}{c} \sin\omega t \quad \frac{\omega\rho}{2c} \qquad \text{inside}$$

(2)

And outside



$$E_\phi \cdot 2\pi\rho = -\frac{1}{c}\partial_t\left(\frac{K_0}{c}\cos\omega t\right)\pi R^2$$

$$E_\phi = \left(\frac{K_0}{c}\sin\omega t\right)\frac{\omega R^2}{2c\rho}$$

Summary

$$B_z = \begin{cases} K_0/c \ \cos\omega t & \text{inside} \\ 0 & \text{outside} \end{cases}$$

$$E_\phi = \begin{cases} K_0/c \ \sin\omega t \left(\frac{\omega\rho}{2c}\right) & \text{inside} \\ \frac{K_0}{c}\sin\omega t \left(\frac{\omega R^2}{2\rho c}\right) & \text{outside} \end{cases}$$

③

b) To determine the $\vec{B}$-field to third order we have (outside the cylinder)

$$\nabla \times B^{(3)} = \frac{1}{c} \partial_t E^{(2)}$$

$$-\frac{\partial B^{z\,(3)}}{\partial \rho} = \frac{1}{c} \frac{K_0}{c} \cos \omega t \left(\frac{\omega^2 R^2}{2c\rho} \right)$$

Integrating From ρ up to ρ_{max}

$$\delta B^z(\rho_{max}) - \delta B^z(\rho) = \frac{K_0}{c} \cos \omega t \left(\frac{\omega^2 R^2}{2c^2} \right) \int_{\rho}^{\rho_{max}} -\frac{d\rho}{\rho}$$

$$= \frac{K_0}{c} \cos \omega t \left(\frac{\omega^2 R^2}{2c^2} \right) (\ln \rho - \ln \rho_{max})$$

To highligth the dependence on ρ we write this (multiplying by -1)

$$\boxed{\delta B^z(\rho) - \delta B^z(\rho_{max}) = \frac{K_0}{c} \cos \omega t \left(\frac{\omega^2 R^2}{2c^2} \right) \left(-\ln \frac{\rho}{R} + \ln \frac{\rho_{max}}{R} \right)}$$

④

c) So we have the quasi-static approx when light traverses the whole system on a time-scale which is much shorter than $\frac{1}{\omega}$.

When ρ becomes large, $\rho_{max} \sim \frac{c}{\omega}$, then the light takes a long time to reach ρ_{max}.

d) Inside the cylinder we have

$$\delta B_z(R) - \delta B_z(\rho) = \int_\rho^R \frac{\partial B^z}{\partial \rho'} d\rho'$$

Ampere's law

$$= \int_\rho^R -\frac{1}{c} \partial_t E^{(2)}(\rho') \, d\rho'$$

$$= \int_\rho^R -\frac{K_0}{c^2} \cos\omega t \; \frac{\omega^2 \rho'}{2c} \; d\rho'$$

$$\delta B_z(R) - \delta B_z(\rho) = -\frac{K_0}{c} \cos\omega t \left(\frac{\omega^2 R^2}{4c^2} - \frac{\omega^2 \rho^2}{4c^2} \right)$$

Multiply by a minus and Adding $B_z(R)$ to both sides (see part (b) for $B_z(R)$)

(5)

We find

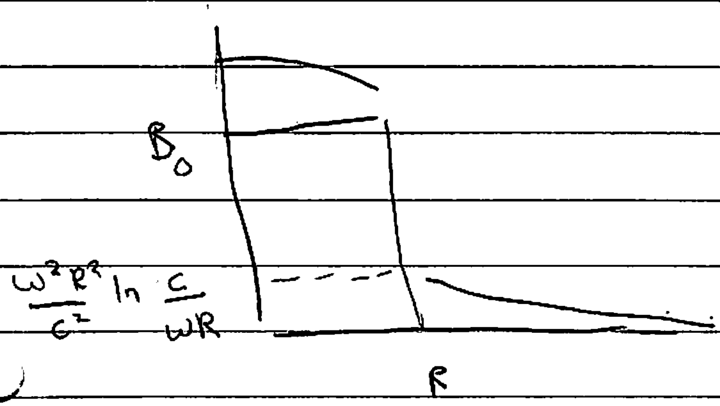
$$\delta B_z(\rho) = \frac{K_o}{c}\cos\omega t \; \frac{\omega^2 R^2}{4c^2}\left(1 - \left(\frac{\rho}{R}\right)^2\right)$$

$$+ \; C$$

Where $C = \frac{K_o}{c}\cos\omega t \left(\frac{\omega^2 R^2}{2c^2}\right) \ln\left(\frac{c}{\omega R}\right) + \text{Order } 1$

Summary and Sketch

$$B_z = \begin{cases} B_z^{\circ} + \frac{K_o}{c}\cos\omega t \; \frac{\omega^2 R^2}{2c^2}\left(\frac{1}{2}\left(1 - \frac{\rho^2}{R^2}\right) + \ln\frac{c}{\omega R}\right) \\ 0 + \frac{K_o}{c}\cos\omega t \; \frac{\omega^2 R^2}{2c^2}\left(-\ln\frac{\rho}{R} + \ln\frac{c}{\omega R}\right) \end{cases}$$



(6) For part a)

$$\vec{A} = \int \frac{\vec{K}/c}{4\pi |\vec{r} - \vec{r}_0|} d^2 r_0$$ ← this is unhelpful

Using:

$$\vec{B} = \nabla \times \vec{A}$$

and noting that $\vec{B} = \frac{K_0}{c} \cos\omega t \ \hat{z}$

Try by guess work $A_\phi = C\rho$ inside

$$(\nabla \times \vec{A})_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (C\rho^2) = 2C = \frac{K_0}{c} \cos\omega t$$

So $C = \frac{K_0}{2c} \cos\omega t$, and

$$\boxed{\vec{A} = \left(\frac{K_0}{2c} \cos\omega t \right) \rho \ \hat{\phi}}$$ ← Coulomb gauge inside

$$\nabla \cdot A = \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} = 0$$

So this is the Coulomb gauge

(7)

In the coulomb gauge

$$-\nabla^2 \varphi = \cancelto{0}{\rho}$$

So $\boxed{\varphi = 0}$ to all orders

Coulomb gauge

The Lorentz gauge we require

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A} = 0$$

So taking $\varphi = 0$ and $\vec{A} = \left(\frac{K_0}{2c}\cos\omega t\right)\rho\,\hat{\phi}$

also satisfies the lorentz gauge condition

So The B and E-fields

$$\boxed{\vec{B} = \nabla\times\vec{A} = \frac{K_0}{c}\cos\omega t\;\hat{z}}$$ (agrees @ before)

And the E-field is

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \cancelto{0}{\vec{\nabla}\varphi}$$

$$\boxed{\vec{E} = +\frac{K_0}{c}\sin\omega t\;\frac{\omega\rho}{2c}}$$ (agrees @ before)

Outside we try motivated by $(\nabla\times A)_z = \frac{1}{\rho}\frac{\partial (A_\phi \rho)}{\partial\rho}$

$$\boxed{A_\phi = \frac{K_o}{2c}\cos\omega t\,\frac{R^2}{\rho}} \longleftarrow \text{Coulomb gauge potential outside}$$

Which is continuous with the inside solution

$$B_z = (\nabla\times A_z) = \frac{1}{\rho}\frac{\partial (A_\phi \rho)}{\partial\rho} = 0$$

The electric field is

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} = \frac{K_o}{c}\sin\omega t\,\frac{\omega R^2}{2c\rho}$$

As before

$$\vec{A}_{LG} = \vec{A}_{\text{Coulomb gauge}}$$

$$\varphi_{LG} = \varphi_{CG} = 0$$

Problem 5. Dipole down the tube (Zangwill)

A small magnet (weight w) falls under gravity down the center of an infinitely long, vertical, conducting tube of radius a, wall thickness $D \ll a$, and conductivity σ. Let the tube be concentric with the $z-$axis and model the magnet as a pointlike dipole with moment $\boldsymbol{m} = m\hat{\mathbf{z}}$. We can find the terminal velocity of the magnet by balancing its weight agains the magnetic drag force associated with the ohmic loss in the walls of the tube.

(a) At the moment it passes through $z = z_o(t)$, show that the magnetic flux produced by $\boldsymbol{m}$ through the a ring of radius a at height z' is

$$\Phi_B = \frac{m}{2}\frac{a^2}{r_o^3} \qquad \text{where} \qquad r_o^2 = a^2 + (z_o - z')^2 \tag{19}$$

(b) When the speed v of the dipole is small, argue that the Faraday EMF induced in the ring is

$$\mathcal{E} = -\frac{1}{c}\partial_t \Phi_B = \frac{v}{c}\frac{\partial \Phi_B}{\partial z'} \tag{20}$$

(c) Show that the current induced in the thin slice of tube which includes the ring is

$$dI = \frac{3mav\sigma D}{4\pi c}\frac{(z_o - z')}{r_o^5}dz' \tag{21}$$

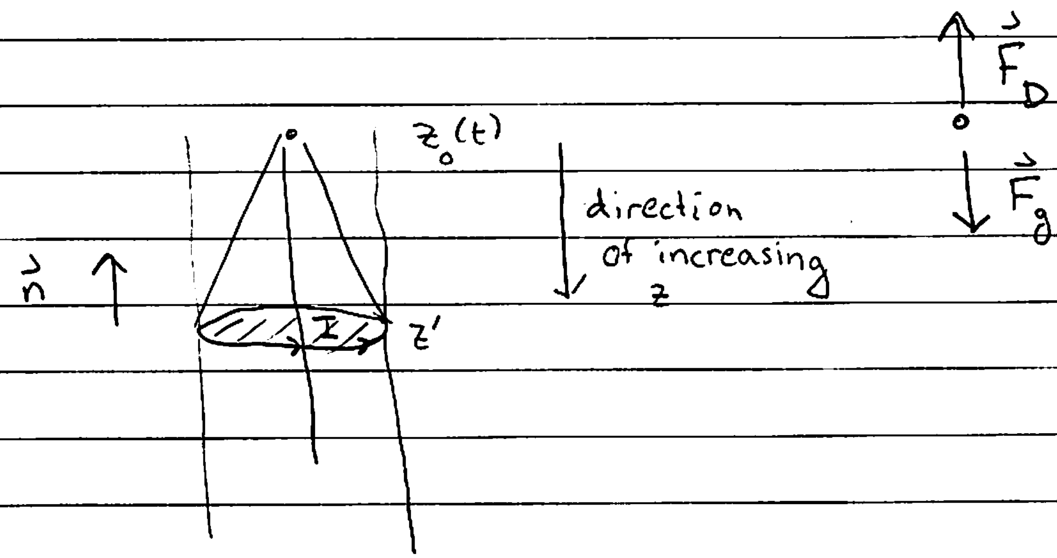
(d) Compute the magnetic drag force $\mathbf{F}$ on $\boldsymbol{m}$ by equating the rate at which the force does work to the power dissipated in the walls of the tube by Joule heating. I find

$$\frac{dE_{\text{ohm}}}{dt} = \sigma D (v/c)^2 \frac{m^2}{a^4}\left(\frac{45}{1024}\right) \tag{22}$$

(e) Find the terminal velocity of the magnet.

① Dipole down the tube:

a)



The dipole falls, inducing currents in the tube, these currents produce a magnetic field which causes a force on the dipole. As magnetic fields do no work, the work done by gravity per time

$$\frac{dW}{dt} = \vec{F}_g(v) \cdot \vec{v}$$

Must be equal to the energy dissipated in the tube

$$\frac{dW}{dt} = \int_{tube} d^3x \; \vec{j}(x) \cdot \vec{E}$$

Here, $\vec{F}_g = -\vec{F}_D$ for a steady state fall

(2)

a) To work out the flux we need to compute:

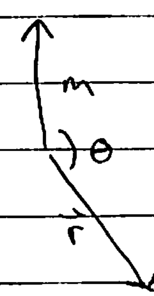
$$\Phi_B(z_0, z') = \int \underbrace{\vec{B}_{dip}}_{\text{mag field due to dipole}} \cdot d\vec{a}_I = \text{magnetic flux through shaded region in figure on previous page}$$

$$= \oint \vec{A}_{dip} \cdot d\vec{l} \leftarrow \text{see loop } I \text{ on previous page}$$

Using

$$\vec{A}_{dip} = \frac{\vec{m} \times \vec{r}}{4\pi r^3} = \frac{m r \sin\theta}{4\pi r^3} \hat{\phi}$$

we have then



$$\Phi_B = \int \underbrace{a\, d\phi}_{dl} \frac{m \sin\theta}{4\pi r^2}$$

$$= \frac{m}{2} \frac{a \sin\theta}{r^2} = \frac{m a^2}{2 r^3}$$

$$\Phi_B = \frac{m a^2}{2} \frac{1}{(a^2 + (z_0(t) - z')^2)^{3/2}}$$

3

b) Then using Maxwell

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

$$-\oint \vec{E}\cdot dl = \frac{1}{c}\partial_t \Phi_B$$

and Symmetry, so that $\oint E\cdot dl = E_\phi\, 2\pi a$

$$-E_\phi\, 2\pi a = \frac{1}{c}\partial_t \Phi_B$$

$$-E_\phi\, 2\pi a = \frac{1}{c}\underbrace{\frac{\partial \Phi}{\partial z_0}}_{\equiv -\frac{\partial \Phi}{\partial z'}}\overbrace{\frac{\partial z_0}{\partial t}}^{v} = -\frac{v}{c}\frac{\partial \Phi}{\partial z'}$$

So $\mathcal{E} = E_\phi\, 2\pi a$ is:

$$\mathcal{E} = +\frac{v}{c}\frac{\partial \Phi}{\partial z'}$$

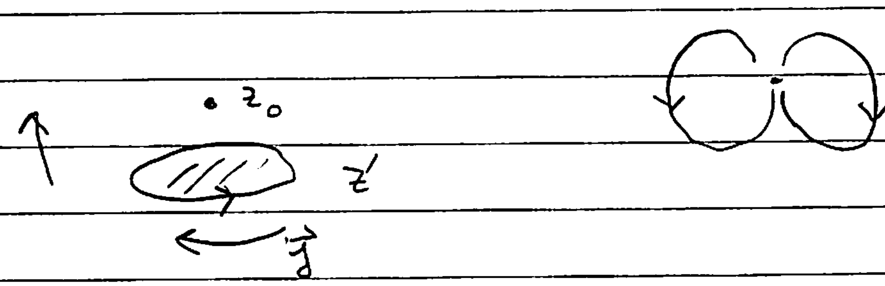
The overall sign depends on the conventions for the loop. This is a consistent set of conventions

c) So the induced current density comes

$$j_\phi = \sigma E_\phi = \frac{1}{2\pi a}\,\sigma\left(\frac{v}{c}\right)\frac{(ma^2/2)}{(a^2+(z_0-z')^2)^{5/2}}\cdot -\frac{3}{2}\cdot 2(z_0-z')(-1)$$

$$j_\phi = \sigma\left(\frac{v}{c}\right)\frac{3ma}{4\pi}\,\frac{(z_0-z')}{(a^2+(z_0-z')^2)^{5/2}}$$

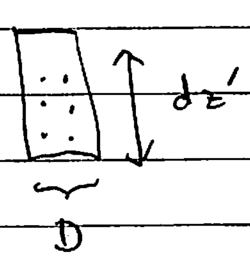
Let us reflect on the signs. When $z' > z_0$ the situation is like this (we use inceasing z being downwards not upwards, see page 1):



B-field of dipole

The dipole flux is getting larger, the current is then in the negative ϕ direction to oppose the change in flux. To find the current, we note

$$\vec{j}\cdot d\vec{A} = \vec{I}\cdot\hat{n}$$



So

$$j_\phi\, D\, dz' = I_\phi$$

5

Then

$$dI_\phi = \frac{3\sigma(V/c)\, maD}{4\pi} \frac{(z_o - z')}{(a^2 + (z_o - z')^2)^{5/2}} dz'$$

d)

We can find the energy dissipated in the system:

$$\frac{dW_{loss}}{dt} \cdot \int_{Vol} \sigma \vec{E} \cdot E \; d^3x$$

$$\frac{dW}{dt} = \sigma D\, 2\pi a \int dz' \, E_\phi^2$$

Using E_ϕ from part c, find

$$\frac{dW}{dt} = \sigma D 2\pi a \left(\frac{V}{c} \frac{3ma}{4\pi}\right)^2 \int dz' \frac{(z_o - z')^2}{(a^2 + (z_o - z')^2)^5}$$

Shifting variables $u = (z' - z_o)/a$, we have

$$\frac{dW}{dt} = \sigma D\, 2\pi a \left(\frac{V}{c} \frac{3ma}{4\pi}\right)^2 \frac{1}{a^7} \underbrace{\int_{-\infty}^{\infty} \frac{u^2}{(1+u^2)^5} du}_{I = \frac{5\pi}{128}}$$

(6)

The integral can be handled by contour integration or elementary methods. Collecting

easiest

$$\frac{dW_{loss}}{dt} = \sigma D \left(\frac{V}{c}\right)^2 \frac{m^2}{a^4} \left(\frac{45}{1024}\right)$$

This work must be equal to $F_D V$, leading to

$$\boxed{F_D = \frac{\sigma D\, V}{c^2} \frac{m^2}{a^4} \left(\frac{45}{1024}\right)}$$

e) So setting $F_D = w$ ← weight

$$\boxed{V = \frac{w}{\frac{\sigma D}{c^2}\frac{m^2}{a^4}} \left(\frac{1024}{45}\right)}$$

Problem 6. Eddy-Current Levitation (Zangwill)

A wire loop of radius b in the $x - y$ plane carries a time-harmonic current $I_o \cos \omega t$. Find the value of I_o needed to levitate a small sphere of mass m, radius a, and conductivity σ at a height z above the center of the loop. Assume $a \ll b$ and that $\delta \ll a$ where δ is the skin depth of the sphere.

(a) First recall in class that we showed that an oscillating magnetic field is damped out over a distance δ. So the picture is that a surface current is generated to satisfy the boundary conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_{\text{out}} - \boldsymbol{H}_{\text{in}}) = \boldsymbol{K} \tag{23}$$

or

$$\boldsymbol{n} \times (\boldsymbol{H}_{\text{out}}) = \boldsymbol{K} \tag{24}$$

since $\boldsymbol{H}_{\text{in}} = 0$. In reality the "surface" current has a thin thickness of order δ, and integrating the current $\boldsymbol{j}$ over the thickness of order δ gives $\boldsymbol{K}$ (as you did in the inclass excercise).

The boundary conditions satisfied by the magnetic field at the surface of the sphere are therefore

$$\boldsymbol{n} \cdot \boldsymbol{B}_{\text{out}} = 0 \tag{25}$$

and

$$\boldsymbol{n} \times \boldsymbol{B}_{\text{out}} = \boldsymbol{K} \tag{26}$$

Solve for the magnetic fields in the vacinity of the sphere (which are affected by the surface currents) with the boundary conditions given above, and the requirement that the field should asymptote to the field of the ring far from the sphere in units of the sphere radius, a. We are, however, still talking about distances very close to the sphere in units the ring radius b, i.e. for

$$a \ll r \ll b \tag{27}$$

the magnetic field approaches the magnetic field of the ring without the sphere at height z.

As an intermediate step show that the vector potential outside the sphere is

$$A_\phi(r, \theta) = \frac{1}{2} B_o r \sin\theta - \frac{B_o a^3}{2r^2} \sin(\theta) \tag{28}$$

using the separation of variables technique. Sketch the lines of magnetic field.

(b) This solution to the fields in the vacinity of the sphere will determine the surface current $\boldsymbol{K}$ and hence the induced magnetic moment on the sphere. Show that

$$\boldsymbol{m} = -\hat{\boldsymbol{z}}\,(2\pi a^3) B_o(z) \tag{29}$$

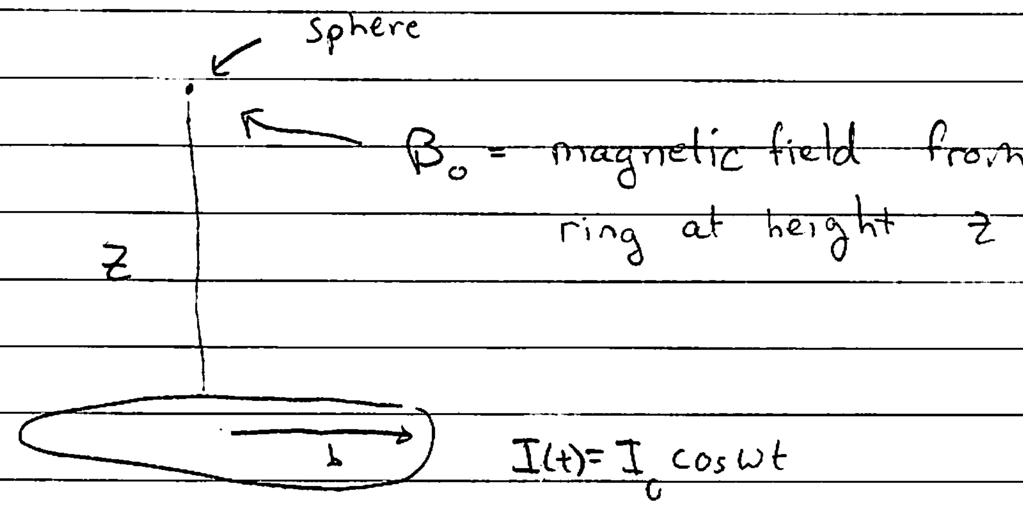
where $B_o(z)$ is the field from the ring at height z. Note that the magnetic moment opposes the applied field.

(c) Then you can compute the force by using the dipole moment and familiar formulas for forces on dipoles in external fields. I find that the time averaged force is

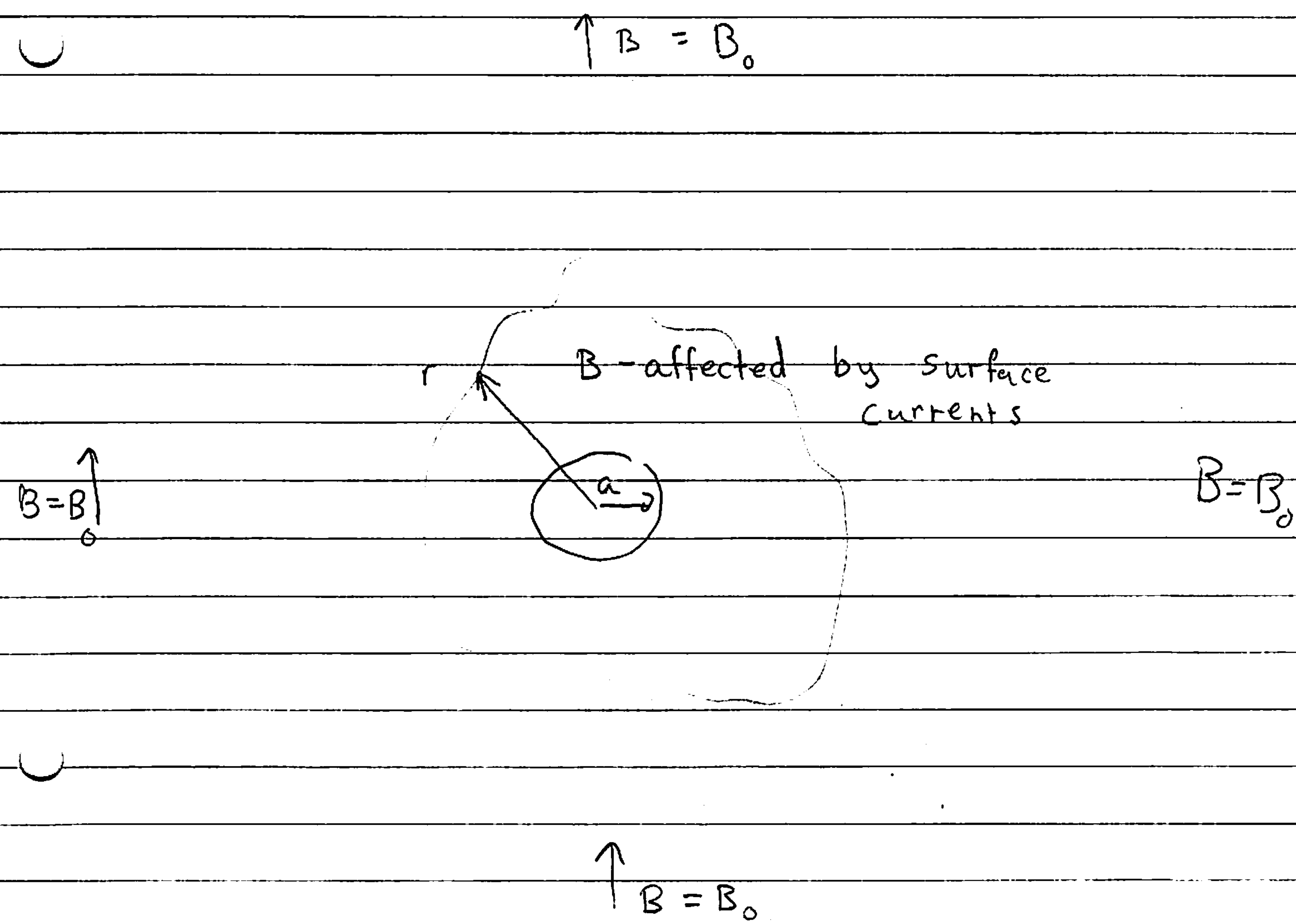
$$\overline{F_z} = \frac{3\pi}{4} \left(\frac{I_o}{c} \right)^2 \frac{a^3 b^4 z}{(z^2 + b^2)^4} = mg \tag{30}$$

①

14.24 - Eddy current levitation



Blow up of Sphere:



②

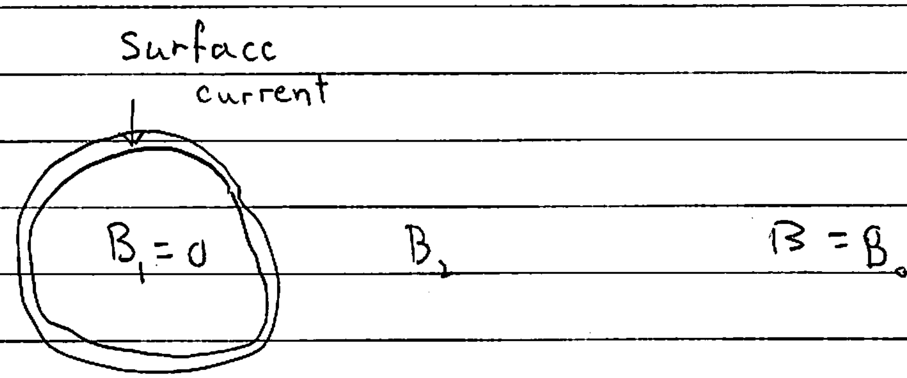
Far from the sphere but much less than z or b

$$a \ll r \ll z$$

We have the mag-field from the ring:

$$B_0 = \frac{I(t)}{c} \frac{\pi b^2}{(z^2+b^2)^{3/2}} \cdot 2 \frac{1}{4\pi} = \frac{I}{2c} \frac{b^2}{(z^2+b^2)^{3/2}}$$

So we now need to solve for $\vec{B}$ in vicinity of spher



The boundary conditions are

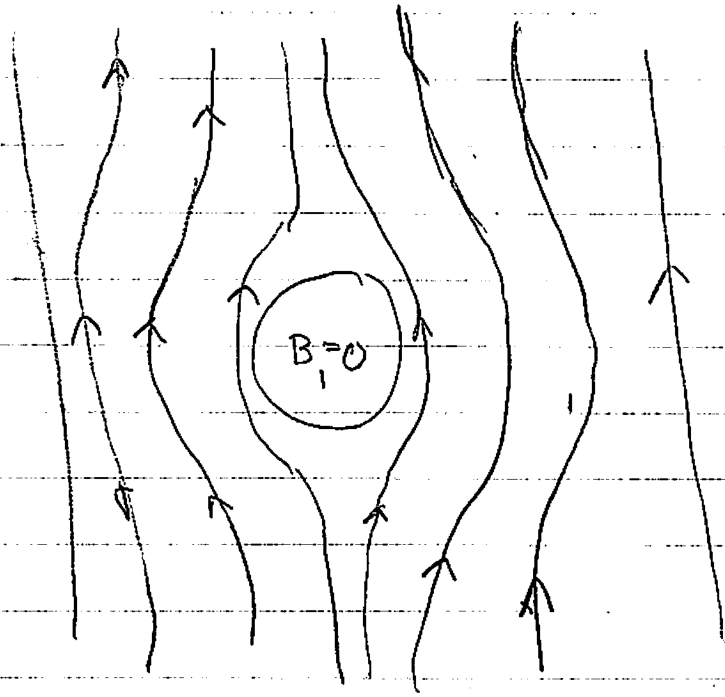
$$\nabla \cdot B = 0 \Rightarrow n \cdot (\vec{B}_2 - \cancelto{0}{\vec{B}_1}) = 0$$

$$\nabla \times \vec{B} = \frac{J}{c} \qquad n \times (\vec{B}_2 - \cancelto{0}{\vec{B}_1}) = \vec{K}/c$$

And we requre $\vec{B}_2 \xrightarrow[r\to\infty]{} \vec{B}_0$

③

So since $\vec{n} \cdot \vec{B}_2 = 0$ The picture is



$B_2 \to B_0 \hat{z}$

Now we will solve for B_2

④

By $r \to \infty$ we mean much greater than a but much less than z

$$a \ll r \ll z$$

So the easiest way is to introduce the magnetic scalar potential

$$\nabla \cdot B = 0$$

$$\nabla \times B = 0 \quad \text{outside Sphere}$$

So $B = -\nabla \Psi_m$ and $-\nabla^2 \Psi_m = 0$

Solving for Ψ_m

$$\Psi_m = \sum_l \left(A_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Demanding

$$B = -\nabla \Psi_m \xrightarrow[r \to \infty]{} B_0 \quad \text{shows} \quad A_1 = -B_0$$

all other A's are zero

$$\Psi_m = -B_0 r\cos\theta + \sum_l \frac{C_l}{r^{l+1}} P_l(\cos\theta)$$

Now from the B.C. $B_r = 0$

$$B_r = -\frac{\partial \Psi_m}{\partial r} = B_o \cos\theta + \sum_l (l+1) \frac{C_l}{r^{l+2}} P_l(\cos\theta) \Big|_{r=a} = 0$$

This is possible if $C_l = 0$ for $l \neq 1$ and

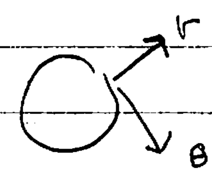
$$B_o + \frac{2C_1}{a^3} = 0 \Rightarrow C_1 = -\frac{B_o a^3}{2}$$

So

$$\Psi_m(\vec{r}) = -B_o r\cos\theta - \frac{B_o a^3 \cos\theta}{2r^2}$$

Then we need:

$$\vec{n} \times \vec{B}_2 = \vec{J}/c$$



$$\hat{r} \times (\hat{r} B_r + \hat{\theta} B_\theta) = \vec{J}/c$$

$$B_\theta \hat{\phi} = \vec{K}/c \leftarrow$$ current points in azimuthal direction

$$B_\theta = -\frac{1}{r}\frac{\partial \Psi_m}{\partial \theta} = -B_o \sin\theta - \frac{B_o a^3}{2r^3}\sin\theta$$

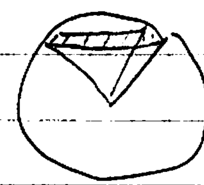
$$B_\theta \Big|_{r=a} = -\frac{3}{2} B_o \sin\theta$$

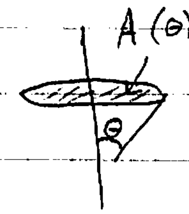
So

$$\vec{K} = -\frac{3}{2} B_o \sin\theta \ \hat{\phi}$$

Now we can evaluate the magnetic moment

$$\vec{m} = \hat{z} \int \frac{dI}{c} \ A(\theta)$$





$$A = \pi a^2 \sin^2\theta \qquad \frac{dI}{c} = a \, d\theta \ K/c$$

$$dI = a \, d\theta \ \left(-\frac{3}{2} B_o \sin\theta\right)$$

So

$$\vec{m} = \hat{z} \int_0^{\pi} a \, d\theta \left(-\frac{3}{2} B_o \sin\theta\right) \pi a^2 \sin^2\theta$$

$$= -\hat{z} \ 3\pi \frac{a^3}{2} B_o \int_0^{\pi} d\theta \ \sin^3\theta$$

$$= -\hat{z} \ \frac{3\pi a^3}{2} B_o \underbrace{\int_{-1}^{1} d\theta \ (1-\cos^2\theta) \ d(\cos\theta)}_{2 - \frac{2}{3} = \frac{4}{3}}$$

$$\vec{m} = -\hat{z} \ 2\pi a^3 B_o$$

⑦

We could have inferred this from

$$\Psi_m = -B_o r\cos\theta + \frac{\vec{m}\cdot\hat{r}}{4\pi r^2} \qquad \vec{m}\cdot\vec{r} = -2\pi a^3 B_o \cos\theta$$

So then

$$F_z = (\vec{m}\cdot\vec{\nabla})B_z = -m\,\partial_z B_z \qquad m \equiv |\vec{m}|$$

With

$$B_o = \frac{I(t)}{2c}\frac{b^2}{(z^2+b^2)^{3/2}}.$$

So $F_z = -2\pi a^3 B_o \partial_z B_o$

$$\boxed{F_z = -\pi a^3 \partial_z B_o^2}$$

$$= \pi a^3 \left(\frac{Ib^2}{2c}\right)^2 -\partial_z \frac{1}{(z^2+b^2)^3}$$

$$\boxed{F_z = (\pi a^3)\left(\frac{I}{c}\right)^2 \frac{\frac{3}{2}b^4 z}{(z^2+b^2)^4} = mg}$$