Problem 1. A conducting slab

A plane polarized electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_I e^{ikz-\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having thickness D. Assume that in space and in the conducting sheet $\mu = \epsilon = 1$, discuss the reflection an transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, corrrect to first order in $(\omega/\sigma)^{1/2}$, are:

$$\frac{E_R}{E_I} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{1}$$

$$\frac{E_T}{E_I} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{2}$$

where

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i) \tag{3}$$

$$\lambda = (1-i)D/\delta \tag{4}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

(b) Verify that for zero thickness and infinite skin depth you obtain the proper limiting results.

(c) **Optional:** Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\mathrm{Re}\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{5}$$

Sketch $\log T$ as a function of D/δ, assuming $\mathrm{Re}\gamma = 10^{-2}$. Define "very small thickness".

Problem 2. In class problems

(a) Consider an incoming plane wave of light propagating in vacuum

$$\boldsymbol{E}_I = E_0 e^{ikz - i\omega t} \boldsymbol{\epsilon} \,. \tag{6}$$

The light is normally incident (i.e. with angle of incidence $\theta_I = 0$) upon a semi-infinite slab of dielectric with $\mu = 1$ and dielectric constant ϵ, which fills the half of space with $z > 0$. Use the reflection and transmission coefficients discussed in class to show that the (time-averaged) force per area on the front face of the dielectric is *away* from the dielectric (i.e. in the $-\hat{\mathbf{z}}$ direction) and is equal in magnitude to

$$\frac{|F^z|}{A} = \frac{1}{2} E_0^2 \left(\frac{n-1}{n+1} \right) \tag{7}$$

(b) Consider an incoming plane wave of light propagating in vacuum

$$\boldsymbol{E}_I = E_0 e^{ikz - i\omega t} \boldsymbol{\epsilon} \,. \tag{8}$$

The light is normally incident upon a slab of metal with conductivity in σ and $\mu = \epsilon = 1$. In class, we evaluated the (time-averaged) Poynting vector just inside the metal and computed the (time-averaged) energy flux into the metal per area per time:

$$\langle \boldsymbol{S} \cdot \boldsymbol{n} \rangle = c \sqrt{\frac{2\omega}{\sigma}} |E_0|^2 \tag{9}$$

Show that this energy flux is equal to (time-averaged) Joule heating in the metal. (Hint: for ohmic material the energy dissipated as heat per volume per time is $\boldsymbol{E} \cdot \boldsymbol{J}$ – I understand this result as $q\boldsymbol{E} \cdot \boldsymbol{v}/\Delta V$ = (force$\times$velcoity)/(Volume).)

Problem 3. Snell's law in a crystal

Consider light of frequency ω in vacuum incident upon a uniform dielectric material filling the space $y > 0$. The light is polarized in plane (as shown below) and has incident angle θ_1. The dielectric material has uniform permittivity ϵ and $\mu = 1$.

(a) Derive Snell's law from the boundary conditions of electrodynamics.

Consider light propagating in a crystal with $\mu = 1$ and dielectric tensor ϵ_{ij}. Along the principal crystalline axes ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \tag{10}$$

and thus, along the axes $D_i = \epsilon_i E_i$ (no sum over i).

(b) Starting directly from the Maxwell equations in the dielectric medium, show that the frequency and wave numbers of the plane wave solutions $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ in the crystal are related by

$$\det\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \epsilon_i}{c^2}\delta_{ij}\right) = 0 \qquad \text{(no sum over } i\text{).} \tag{11}$$

Now consider light of frequency ω in vacuum incident upon a dielectric crystal. The light has incident angle θ_1, and propagates in the $x - y$ plane, *i.e.* $k_z = 0$. The incident light is also polarized in $x - y$ plane, and the axes of the dielectric crystal are aligned with the x, y, z axes (see below). Only the y axis of the crystal has a slightly larger dielectric constant than the remaining two axes,

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon\,(1+\delta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{12}$$

with $\delta \ll 1$.

(c) Determine angle of refraction (or $\sin\theta_2$) including the first order in δ correction to Snell's law.

(d) Is the refracted angle smaller or larger than in the isotropic case? Explain physically. Does the angular dependence of your correction makes physical sense? Explain physically.

(e) If the incident light is polarized along the z axis (out of the $x - y$ plane), what is the deviation from Snell's law? Explain physically.

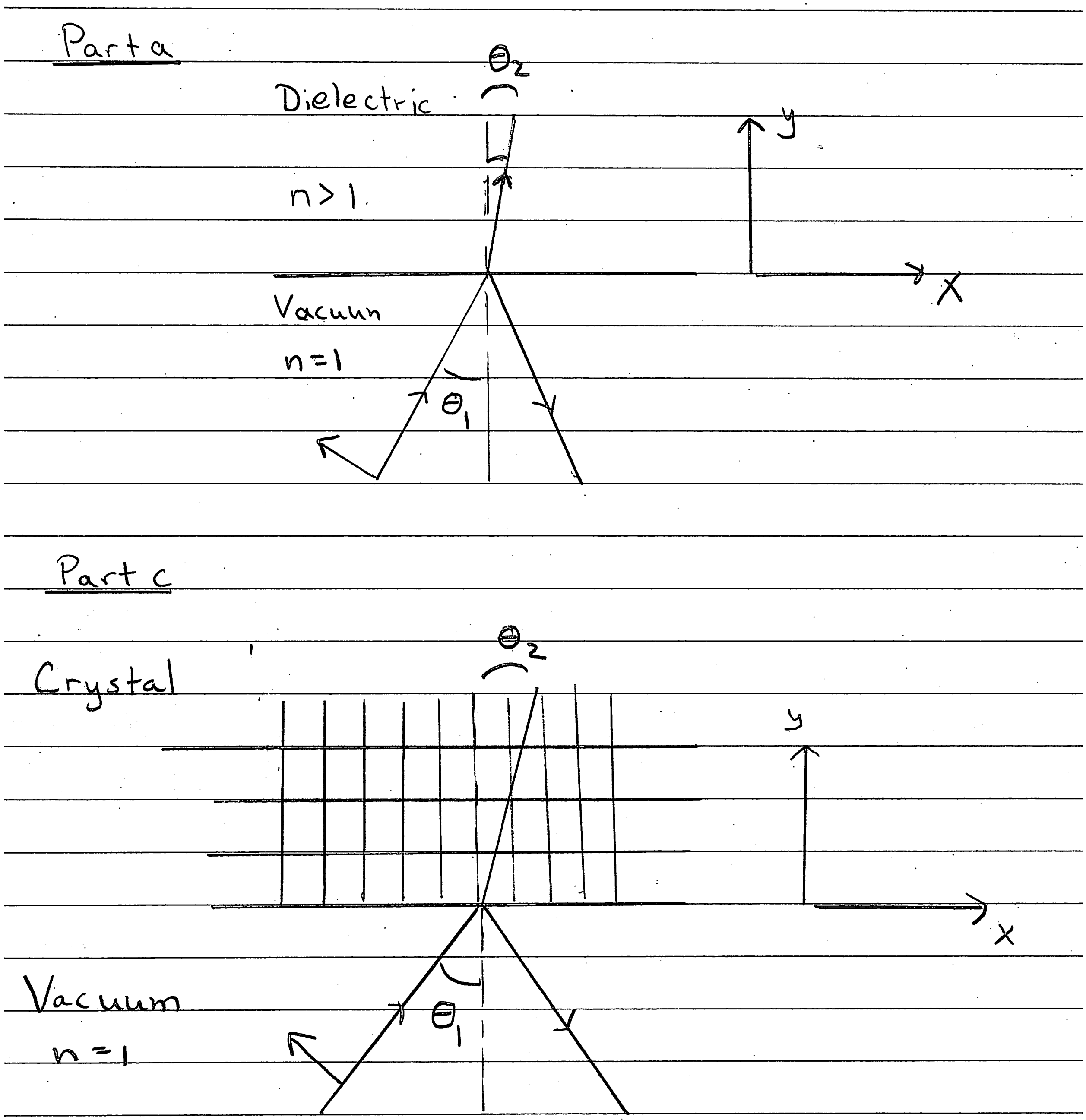
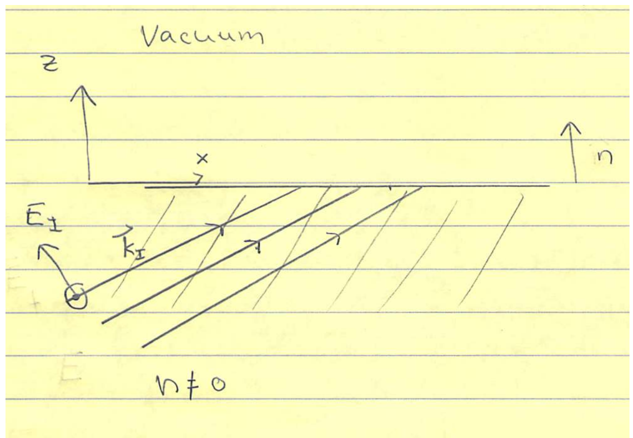


Figure 1: Snell's law geometry

Problem 4. Analysis of the Good-Hänchen effect

A "ribbon" beam[1] of in plane polarized radiation of wavelength λ is totally internally reflected at a plane boundary between a non-permeable (i.e. $\mu = 1$) dielectric media with index of refraction n and vacuum (see below). The critical angle for total internal reflection is θ_I^o, where $\sin\theta_I^o = 1/n$. First assume that the incident wave takes the form $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}_I e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ of a pure plane wave polarized in plane and study the transmitted and reflected waves.



(a) Starting from the Maxwell equations, show that for $z > 0$ (i.e. in vacuum) the electric field takes the form:

$$\boldsymbol{E}_2(x, z) = \boldsymbol{E}_2 e^{-\frac{\omega}{c}(\sqrt{n^2 \sin\theta_I^2 - 1})z} e^{i\frac{\omega n \sin\theta_I}{c} x} \tag{13}$$

(b) Starting from the Maxwell equations, show that for $\theta_I > \theta_I^0$ the ratio of the reflected amplitude to the incident amplitude is a pure phase

$$\frac{E_R}{E_I} = e^{i\phi(\theta_I, \theta_I^o)} \tag{14}$$

and determine the phase angle. Thus the reflection coefficient $R = |E_R/E_I|^2 = 1$ However, phase has consequences.

(c) Show that for a monochromatic (*i.e.* constant $\omega = ck$) ribbon beam of radiation in the z direction with a transverse electric field amplitude, $E(x)e^{ik_z z - i\omega t}$, where $E(x)$ is smooth and finite in the transverse extent (but many wavelengths broad), the lowest order approximation in terms of plane waves is

$$\boldsymbol{E}(x, z, t) = \boldsymbol{\epsilon} \int \frac{d\kappa}{(2\pi)} A(\kappa) e^{i\kappa x + ikz - i\omega t} \tag{15}$$

where $k = \omega/c$. Thus, the finite beam consists of a sum plane waves with a small range of angles of incidence, centered around the geometrical optics value.

[1] By a "ribbon" beam I mean a beam which has finite transverse extent in the direction perperndicular $\boldsymbol{k}_I$ lying in the x-z place as drawn above. But, the beam is infinite in extent in the y direction (coming out of page in the figure above) . Thus the incoming and outgoing "ribbion" beams form a kind of wedge.

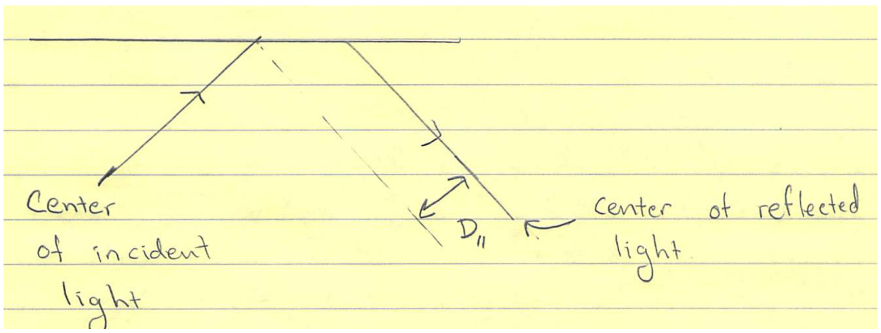
(d) Consider a reflected ribbon beam and show that for $\theta_I > \theta_I^o$ the electric field can be expressed approximately as

$$\boldsymbol{E}_R = \boldsymbol{\epsilon}_R E(x'' - \delta x) e^{i\boldsymbol{k}_R\cdot\boldsymbol{r} - i\omega t + i\phi(\theta_I, \theta_I^o)} \tag{16}$$

where $\boldsymbol{\epsilon}_R$ is a polarization vector, x'' is the coordinate perpendicular to the reflected wave vector $\boldsymbol{k}_R$, and the displacement $\delta x = -\frac{1}{k}\frac{d\phi}{d\theta_I}$ is determined by phase shift.

(e) Using the phase shift you computed, show that the lateral shift of the reflected in plane polarized beam is

$$D_\parallel = \frac{\lambda}{\pi} \frac{\sin\theta_I}{\sqrt{\sin^2\theta_I - \sin^2\theta_I^o}} \frac{\sin^2\theta_I^o}{\sin\theta_I^2 - \cos\theta_I^2 \sin^2\theta_I^o} \tag{17}$$



Problem 5. Reflection of a Gaussian Wave Packet Off a Metal Surface:

In class we showed that the amplitude reflection coefficient from a good conductor ($\omega \ll \sigma$) for a plane wave of wavenumber $k = \omega/c$ is

$$\frac{H_R(k)}{H_I(k)} = 1 - \sqrt{\frac{2\mu\omega}{\sigma}}(1-i) \simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi(\omega)}\,, \tag{18}$$

where the phase is for $\omega \ll \sigma$:

$$\phi(\omega) \simeq \sqrt{\frac{2\mu\omega}{\sigma}}\,. \tag{19}$$

Consider a Gaussian wave packet with average wave number k_o centered at $z = -L$ at time $t = -L/c$ which travels towards a metal plane located at $z = 0$ and reflects. Show that the time at which the center of the packet returns to $z = -L$ is given by

$$t = \frac{L}{c} + \frac{\mu\delta_o}{2c} \tag{20}$$

where the time delay is due to the phase shift $d\phi(\omega_o)/d\omega$, and $\delta_o = \sqrt{2c/\sigma\mu k_o}$ is the skin depth.