Problem 1. A conducting slab

A plane polarized electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_I e^{ikz-\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having thickness D. Assume that in space and in the conducting sheet $\mu = \epsilon = 1$, discuss the reflection an transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, corrrect to first order in $(\omega/\sigma)^{1/2}$, are:

$$\frac{E_R}{E_I} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{1}$$

$$\frac{E_T}{E_I} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{2}$$

where

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i) \tag{3}$$

$$\lambda = (1-i)D/\delta \tag{4}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

(b) Verify that for zero thickness and infinite skin depth you obtain the proper limiting results.

(c) **Optional:** Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\mathrm{Re}\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{5}$$

Sketch $\log T$ as a function of D/δ, assuming $\mathrm{Re}\gamma = 10^{-2}$. Define "very small thickness".

Problem 1. A conducting slab

A plane polarized electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_I e^{ikz-\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having thickness D. Assume that in space and in the conducting sheet $\mu = \epsilon = 1$, discuss the reflection an transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, corrrect to first order in $(\omega/\sigma)^{1/2}$, are:

$$\frac{E_R}{E_I} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{1}$$

$$\frac{E_T}{E_I} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{2}$$

where

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i) \tag{3}$$

$$\lambda = (1-i)D/\delta \tag{4}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

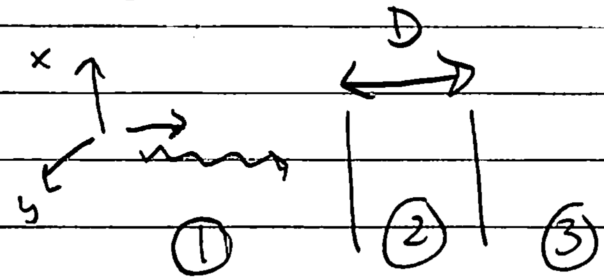
(b) Verify that for zero thickness and infinite skin depth you obtain the proper limiting results.

(c) **Optional:** Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\mathrm{Re}\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{5}$$

Sketch $\log T$ as a function of D/δ, assuming $\mathrm{Re}\gamma = 10^{-2}$. Define "very small thickness".

Problem 1



$$\left.\begin{aligned} E_{①} &= E_I\, e^{ikz - i\omega t} + E_R\, e^{-ikz - i\omega t} \quad \hat{x} \\ H_{①} &= E_I\, e^{ikz - i\omega t} - E_R\, e^{-ikz - i\omega t} \quad \hat{y} \end{aligned}\right\} \quad \text{Eq. 1}$$

- We used $\frac{i\omega}{c}\vec{B} = i\vec{k} \times \vec{E}$ to write & recongnize that $\vec{B}$ is orthogonal to $\vec{E}$ and $\vec{k}$, but is of equal magnitude.

- In Region 2 we have two plane waves solutions specified by $k_\pm = \pm(1+i)/\delta$ where

$$\delta = \sqrt{\frac{2c^2}{\cancel{\mu}\sigma\omega}} = \sqrt{\frac{2c^2}{\sigma\omega}}$$

Then

$$H_{②} = \underbrace{H_+ e^{ik_+ z - i\omega t}}_{\text{expontially growing}} + \underbrace{H_- e^{-ik_+ z - i\omega t}}_{\text{expontially shrinking}} \qquad (\text{Eq. 2a})$$

(2) ...

Using

$$i\vec{k} \times \vec{B} = \sigma \vec{E} \Rightarrow \frac{i\vec{k}}{\sigma} \times \vec{B}$$

Taking $\vec{B}$ in the y-direction lets look at the first wave (+)

$$\vec{H}_+ = H_+ e^{ik_+ \hat{z} \cdot \vec{r}}$$

So

$$\vec{E}_+ = \frac{ik_+}{\sigma} \hat{z} \times H_+ \hat{y}$$

$$\vec{E}_+ = -\frac{ik_+}{\sigma} H_+ \hat{x} = \frac{(1-i)}{\delta\sigma} H_+$$

And similarly

$$\vec{H}_- = H_- e^{ik_+ (-\hat{z}) \cdot \vec{r} - i\omega t}$$

So $$\vec{E}_- = -\frac{(1-i)}{\delta\sigma} H_- \hat{x}$$

So

$$E_{(2)} = \left[\frac{(1-i)}{\delta\sigma} H_+ e^{+ik_+ z - i\omega t} - \frac{(1-i)}{\delta\sigma} H_- e^{-ik_+ z - i\omega t} \right] \hat{x}$$

(Eq. 2b)

Finally we examine region (3)

③

$$\left.\begin{aligned} E_{③} &= E_T\, e^{ikz-i\omega t}\ \hat{x} \\ H_{③} &= {}'E_T\, e^{ikz-i\omega t}\ \hat{y} \end{aligned}\right\} \quad \text{Eq (3)}$$

In summary, in Eqs. (1), (2), (3) we have written
The fields through the scalar amplitudes:

$$\underset{\text{inputs}}{E_I}, \underbrace{E_R, H_+, H_-, E_T}_{\text{what we need to find}}$$

So we need four boundary condition which come from

$\nabla \cdot D = \rho$ $\qquad n \cdot (D_2 - D_1) = \Sigma$ ← normal eq

$\nabla \times H = \frac{J}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$ $\qquad n \times (H_2 - H_1) = K$ ← transverse eq

$\nabla \cdot B = 0$ $\qquad n \cdot (B_2 - B_1) = 0$ ← normal eq.

$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$ $\qquad n \times (E_2 - E_1) = 0$ ← transv eq

Now the normal equation given nothing useful
Since the way we parametrized our fields
$n \cdot \vec{E} = \vec{n} \cdot \vec{B} = 0$

④

There are no surface currents or charges so Σ and $K = 0$. And the transverse equations just give continuity of $E^{\perp}$ and $H^{\perp}$

- Continuity of E and H at interface $\underline{1}$ $(z=0)$ gives

1) $$E_I + E_R = \frac{(1-i)}{\delta\sigma} H_+ - \frac{(1-i)}{\delta\sigma} H_- \qquad (E\ \text{cont})$$

2) $$E_I - E_R = H_+ + H_- \qquad (H\text{-cont})$$

- Continuity of E and H at interface 2 $(z=D)$ give

3) $$H_+ e^{ik_+D} + H_- e^{-ik_+D} = E_T e^{ikD} \qquad (H\text{-cont})$$

4) $$\frac{(1-i)}{\delta\sigma} H_+ e^{ik_+D} - \frac{(1-i)}{\delta\sigma} H_- e^{-ik_+D} = E_T e^{ikD} \qquad (E\text{-cont})$$

1), 2), 3), 4) give sufficient info to solve for E_T, E_R, H_+, H_-

(5)

The equations read, defining $\tilde{E}_T = E_T e^{ikD}$:

$$E_I + E_R = \frac{\gamma}{2} H_+ - \frac{\gamma}{2} H_-$$

$$E_I - E_R = H_+ + H_-$$

$$H_+ e^{-2\lambda} + H_- = \tilde{E}_T e^{-\lambda}$$

$$\frac{\gamma}{2} H_+ e^{-2\lambda} - \frac{\gamma}{2} H_- = \tilde{E}_T e^{-\lambda}$$

So:

$$E_{(3)} = \tilde{E}_T e^{ik(x-D)-i\omega t} = E_T e^{ikx-i\omega t}$$

$$-\lambda \equiv i k_+ D = -\frac{D}{\delta}(1-i)$$

Solving and setting $\gamma^2 \to 0$ as it is order $\frac{1}{\sigma}$, i.e. γ is small

$$\frac{E_R}{E_I} = -\frac{(1-e^{-2\lambda})}{(-1+\gamma)e^{-2\lambda} + (1+\gamma)}$$

$$\frac{E_T}{E_I} = +\frac{2e^{-\lambda}\gamma}{(-1+\gamma)e^{-2\lambda} + (1+\gamma)}$$

Here

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i)$$

$$\lambda = (1-i)D/\delta$$

(6)

b) For $D \to 0$ $\lambda \to 0$

$$E_R \propto (1 - e^{-2\lambda}) \longrightarrow 0$$

$$\frac{E_T}{E_I} = \frac{2\gamma}{(-1+\gamma)+(1+\gamma)} = 1$$

c) $$\frac{|E_T|^2}{|E_I|^2} = \frac{\left|2\, e^{-D/\delta}\, \sqrt{2}\, \mathrm{Re}\,\gamma\ e^{i\phi_1}\, e^{i\phi_2}\right|^2}{\left|(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})\right|^2}$$

We wrote:

$$e^{-\lambda} = e^{-D/\delta}\, e^{iD/\delta} = e^{-D/\delta}\, e^{i\phi_1}$$

and

$$\gamma = \frac{\omega\delta}{2}(1-i) = \sqrt{2}\,\frac{\omega\delta}{2}\,\frac{(1-i)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\gamma\ e^{i\phi_2}$$

$$|E_T|^2 = \frac{8(\mathrm{Re}\,\gamma)^2 e^{-2D/\delta}}{|1-e^{-2\lambda}|^2 + O(\gamma)} \leftarrow \text{dropped}$$

$$|1-e^{-2\lambda}|^2 = 1 + e^{-2\lambda} e^{-2\lambda^*} - 2\,\mathrm{Re}\, e^{-2\lambda}$$

$$= 1 + e^{-4D/\delta} - 2\, e^{-2D/\delta}\, \mathrm{Re}\, e^{2iD/\delta}$$

$$= 1 + e^{-4D/\delta} - 2\, e^{-2D/\delta} \cos(2D/\delta)$$

⑦

So

$$\frac{|E_T|^2}{|E_I|^2} = \frac{8 \, \mathrm{Re}\,\gamma \, e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}}$$

This approx is good, provided the $O(\gamma)$ term dropped is small compared to the denom that was kept.

Expanding

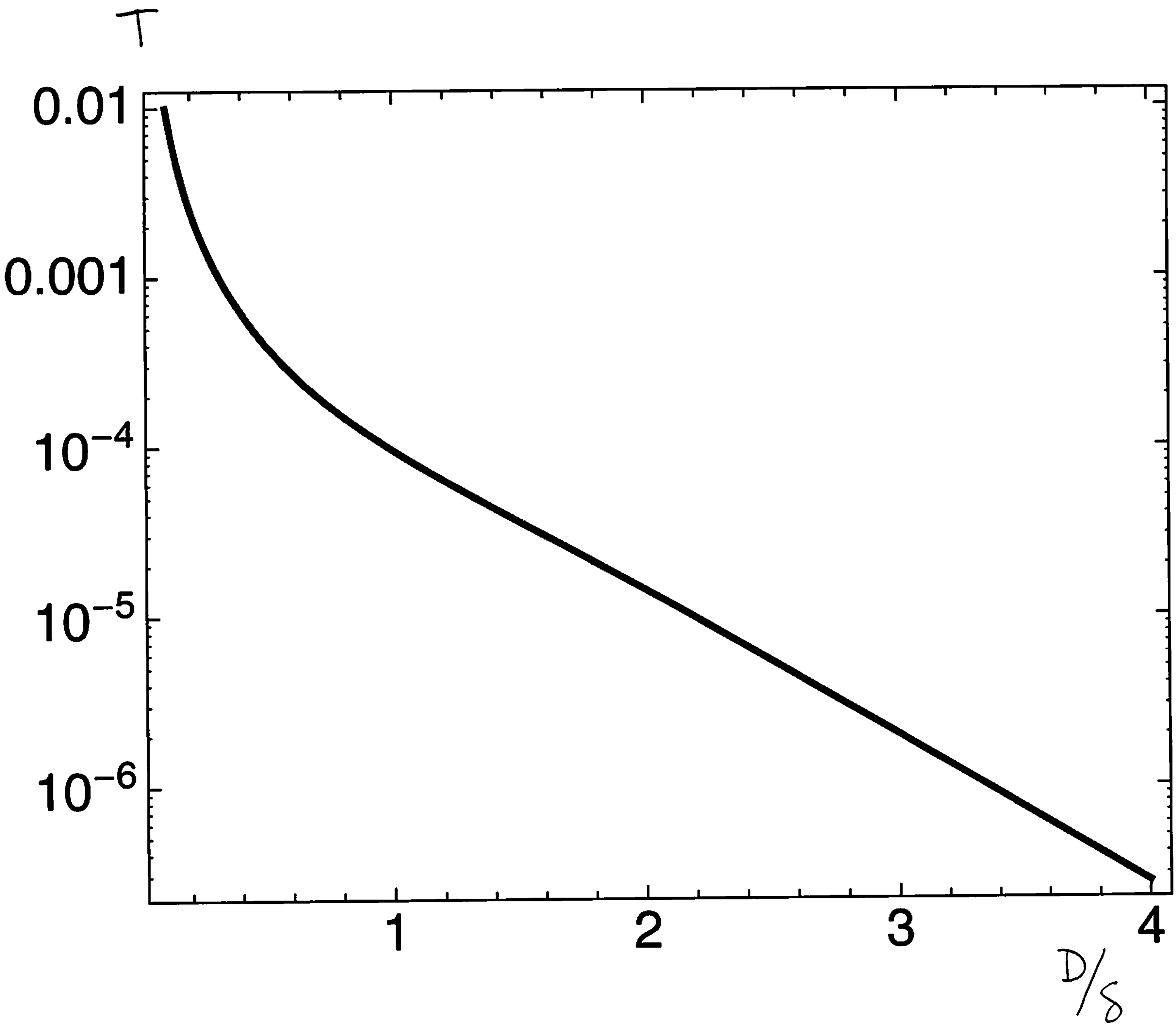
$$1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}$$

$$\simeq 8\left(\frac{D}{\delta}\right)^2 + O\left(\left(\frac{D}{\delta}\right)^4\right)$$

So we need

$$8\left(\frac{D}{\delta}\right)^2 \sim \gamma^2$$

$$\boxed{\frac{D}{\delta} \gg \gamma}$$


T
0.01
0.001
10^{-4}
10^{-5}
10^{-6}
1
2
3
4
D/δ

Problem 2. In class problems

(a) Consider an incoming plane wave of light propagating in vacuum

$$\boldsymbol{E}_I = E_0 e^{ikz - i\omega t} \boldsymbol{\epsilon} \,. \tag{6}$$

The light is normally incident (i.e. with angle of incidence $\theta_I = 0$) upon a semi-infinite slab of dielectric with $\mu = 1$ and dielectric constant ϵ, which fills the half of space with $z > 0$. Use the reflection and transmission coefficients discussed in class to show that the (time-averaged) force per area on the front face of the dielectric is *away* from the dielectric (i.e. in the $-\hat{\mathbf{z}}$ direction) and is equal in magnitude to

$$\frac{|F^z|}{A} = \frac{1}{2} E_0^2 \left(\frac{n-1}{n+1} \right) \tag{7}$$

(b) Consider an incoming plane wave of light propagating in vacuum

$$\boldsymbol{E}_I = E_0 e^{ikz - i\omega t} \boldsymbol{\epsilon} \,. \tag{8}$$

The light is normally incident upon a slab of metal with conductivity in σ and $\mu = \epsilon = 1$. In class, we evaluated the (time-averaged) Poynting vector just inside the metal and computed the (time-averaged) energy flux into the metal per area per time:

$$\langle \boldsymbol{S} \cdot \boldsymbol{n} \rangle = c \sqrt{\frac{2\omega}{\sigma}} |E_0|^2 \tag{9}$$

Show that this energy flux is equal to (time-averaged) Joule heating in the metal. (Hint: for ohmic material the energy dissipated as heat per volume per time is $\boldsymbol{E} \cdot \boldsymbol{J}$ – I understand this result as $q\boldsymbol{E} \cdot \boldsymbol{v}/\Delta V$ = (force×velcoity)/(Volume).)

Energy dissipated in Conductor - Direct Calculation pg. 1

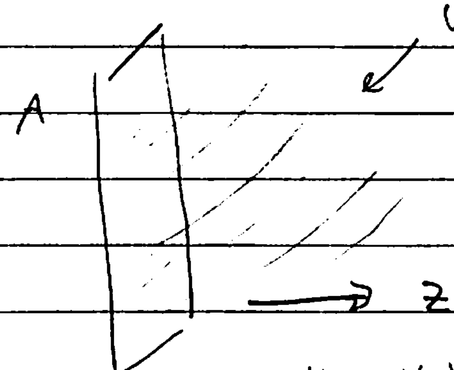
$$\left\langle \frac{dW}{dt} \right\rangle_{\substack{\text{time}\\\text{ave}}} = \int_V d^3x \, \frac{Re}{2}(\vec{j}^* \cdot \vec{E})$$

for a time ave take half Re part

$j = \sigma E$

$$= \frac{1}{2}\sigma \int_V d^3x \, |E|^2$$

$$= \frac{1}{2}\sigma \underbrace{A}_{\text{area}} \int_0^\infty dz \, |E^2|$$



Using

H_c Value of H at interface $\simeq 2H_I$

$$|E_c|^2 = \frac{\mu\omega}{\sigma} |H_c|^2 e^{-2z/\delta}$$

do integral use, $\delta = \sqrt{\frac{2c^2}{\mu\sigma\omega}}$,

So

$$\boxed{\frac{1}{A}\left\langle \frac{dW}{dt} \right\rangle} = \frac{1}{2}\sigma \int_0^\infty dz \, \frac{\mu\omega}{\sigma} 4H_I^2 e^{-2z/\delta} = \boxed{c\sqrt{\frac{2\mu\omega}{\sigma}} \, |H_I|^2}$$

This agrees @ S_{loss} the flux into metal.

Problem 3. Snell's law in a crystal

Consider light of frequency ω in vacuum incident upon a uniform dielectric material filling the space $y > 0$. The light is polarized in plane (as shown below) and has incident angle θ_1. The dielectric material has uniform permittivity ϵ and $\mu = 1$.

(a) Derive Snell's law from the boundary conditions of electrodynamics.

Consider light propagating in a crystal with $\mu = 1$ and dielectric tensor ϵ_{ij}. Along the principal crystalline axes ϵ_{ij} is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \tag{10}$$

and thus, along the axes $D_i = \epsilon_i E_i$ (no sum over i).

(b) Starting directly from the Maxwell equations in the dielectric medium, show that the frequency and wave numbers of the plane wave solutions $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ in the crystal are related by

$$\det\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \epsilon_i}{c^2} \delta_{ij}\right) = 0 \qquad \text{(no sum over } i\text{)}. \tag{11}$$

Now consider light of frequency ω in vacuum incident upon a dielectric crystal. The light has incident angle θ_1, and propagates in the $x - y$ plane, *i.e.* $k_z = 0$. The incident light is also polarized in $x - y$ plane, and the axes of the dielectric crystal are aligned with the x, y, z axes (see below). Only the y axis of the crystal has a slightly larger dielectric constant than the remaining two axes,

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon\,(1+\delta) & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{12}$$

with $\delta \ll 1$.

(c) Determine angle of refraction (or $\sin\theta_2$) including the first order in δ correction to Snell's law.

(d) Is the refracted angle smaller or larger than in the isotropic case? Explain physically. Does the angular dependence of your correction makes physical sense? Explain physically.

(e) If the incident light is polarized along the z axis (out of the $x - y$ plane), what is the deviation from Snell's law? Explain physically.

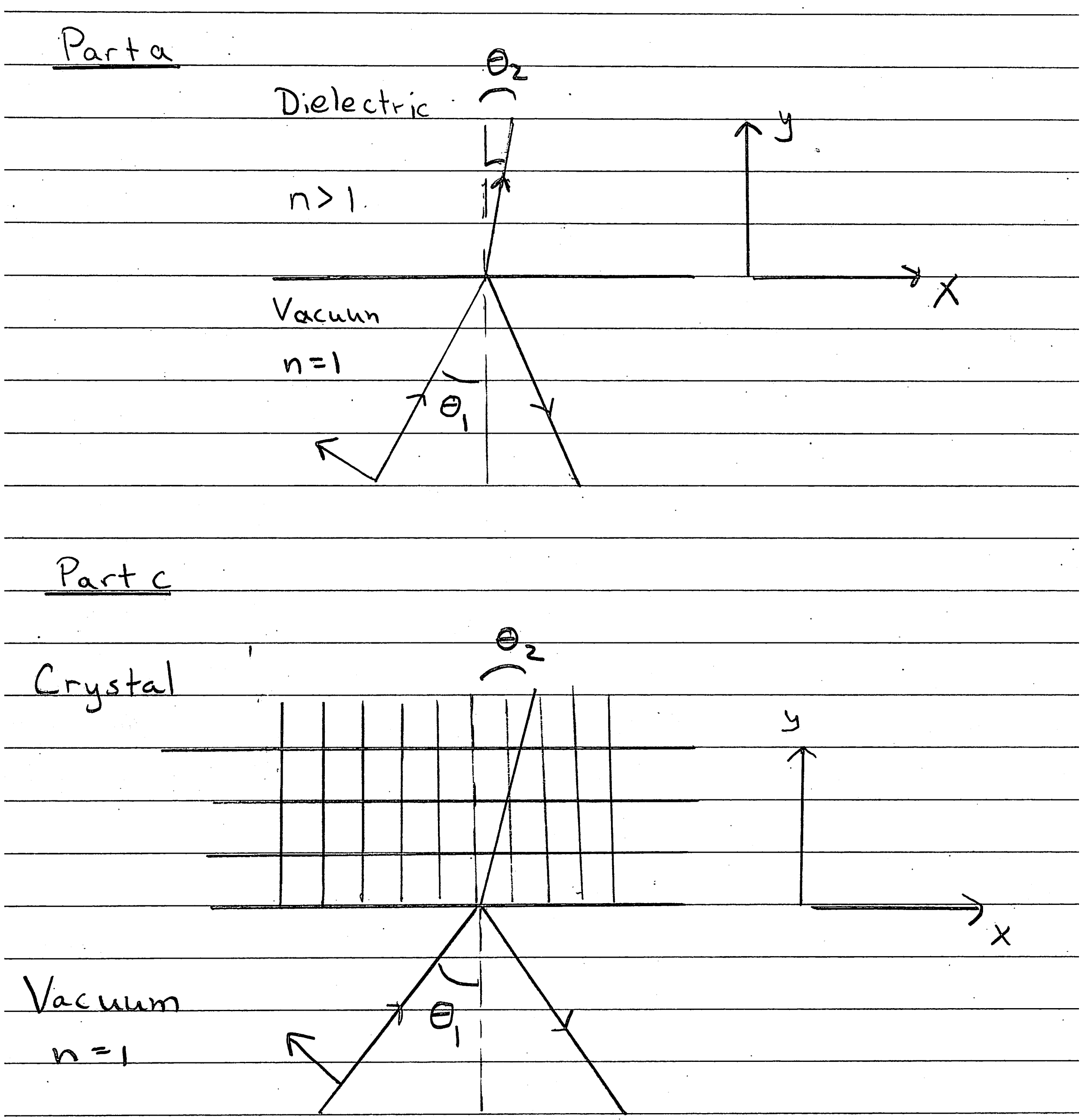
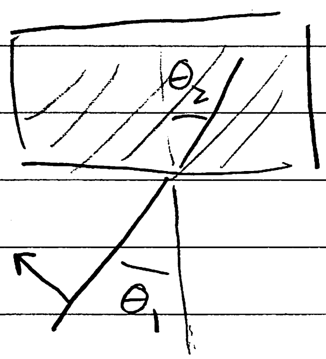


Figure 1: Snell's law geometry

Snell's Law pg. 1

a)



The incident plane wave with $|\vec{k}_R| = |\vec{k}_I| = \omega/c$ has:

$$\vec{E} = \vec{E}_I e^{i\vec{k}_I \cdot \vec{r} - i\omega t} + \vec{E}_R e^{i\vec{k}_R \cdot \vec{r} - i\omega t}$$

Then the transmitted wave:

$$\vec{E} = E_T e^{i\vec{k}_T \cdot \vec{r} - i\omega t} \qquad |\vec{k}_T| = n_2 \frac{\omega}{c}$$

The equality of phases at interface, is needed for b.c:

$$i\vec{k}_I \cdot \vec{r} - i\omega t \Big|_{y=0} = +i\vec{k}_T \cdot \vec{r} - i\omega t \Big|_{y=0}$$

gives:

$$ik_I^x \Big| = ik_T^x \Big| \qquad -i\omega_I = -i\omega_T$$

So

$$\sin\theta_1 = \frac{k_x}{k_I} \qquad \sin\theta_2 = \frac{k_T^x}{k_T}$$

$$\sin\theta_1 = \frac{ck_x}{\omega} \qquad \sin\theta_2 = \frac{c}{n\omega} \Rightarrow \boxed{\frac{1}{n_2}\sin\theta_1 = \sin\theta_2}$$

Snell's Law pg. 2

b) $\nabla \cdot D = \rho$

$\nabla \times B = J/c + \frac{1}{c}\partial_t D$

$\nabla \cdot B = 0$

$-\nabla \times E = \frac{1}{c}\partial_t B$

In free space $\rho = J/c = 0$ then for plane wave e.g.

$$\vec{E} = \vec{E}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

find:

$ik \cdot D = 0$

$i\vec{k} \times \vec{B} = -\frac{i\omega}{c} D$

$i\vec{k} \cdot \vec{B} = 0$

$-ik \times \vec{E} = -\frac{i\omega}{c} B$

So

$ik \times (-i\vec{k} \times \vec{E}) = -\frac{i\omega}{c} i\vec{k} \times B$

$\vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E} = -\frac{i\omega}{c}\left(-\frac{i\omega}{c}\vec{D}\right)$

Snell's Law pg. 3

So we have with $D_i = \varepsilon_i E_i$ (no i sum)
that

$$k_i (k_j E_j) - k^2 E_i = -\frac{\omega^2}{V_i^2} E_i \qquad V_i^2 \equiv \frac{c^2}{\varepsilon_i}$$

Or

$$\left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2}{V_i^2} \delta_{ij} \right) E_j = 0 \qquad (\text{Eq} \star)$$

This will only have non-trivial solutions if the determinant is non-zero, by the theory of linear equations:

$$\det \left(k_i k_j - k^2 \delta_{ij} + \frac{\omega^2 \delta_{ij}}{V_i^2} \right) = 0$$

c) The angle of refraction is derived following the methods part a). From the equality of phases

$$k^x = k^x_I = k^x_T \qquad \omega_I = \omega_R = \omega$$

Now we can write out Eq ☆ :

$$\begin{pmatrix} -k_y^2 + \frac{\omega^2}{V_x^2} & k_x k_y & 0 \\ k_x k_y & -k_x^2 + \frac{\omega^2}{V_x^2} & 0 \\ 0 & 0 & -k^2 + \frac{\omega^2}{V_x^2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = ($$

Taking the determinant :

$$D = \left(-k^2 + \frac{\omega^2}{V_x^2}\right)\left(\left(-k_y^2 + \frac{\omega^2}{V_x^2}\right)\left(-k_x^2 + \frac{\omega^2}{V_y^2}\right) - k_x^2 k_y^2\right)$$

$$D = \left(-k^2 + \frac{\omega^2}{V_x^2}\right)\left(-\frac{\omega^2 k_x^2}{V_x^2} - \frac{\omega^2 k_y^2}{V_y^2} + \frac{\omega^4}{V_x^2 V_y^2}\right)$$

Setting $D = 0$ we have two solutions :

$$\left(-k^2 + \frac{\omega^2}{V_x^2}\right) = 0$$

← we will return to this in part e). The eigenvector $\vec{E}_T$ is polarized in the z direction. See matrix

Snell's Law pg. 4½

and

$$\left(-\frac{\omega^2}{V_x^2} - \frac{\omega^2}{V_y^2} k_y^2 + \frac{\omega^4}{V_x^2 V_y^2} \right) = 0$$

In this case we must set $E_z = 0$ (see matrix) and the eigenvector is polarized in plane

Snell's Law pg. 5

writing $V_x^2 = \frac{c^2}{\varepsilon}$, $V_y^2 = \frac{c^2}{\varepsilon(1+\delta)}$ and solving

for k_y from the second term in braces:

$$(k^y)^2 = \frac{\omega^2}{V_x^2} - \frac{V_y^2 k_x^2}{V_x^2}$$

now we know K_x and k_y so we know the angle $\tan\theta_R = \frac{k^y}{k^x}$

The rest is algebra

Writing

$$(k^y)^2 = \frac{\omega^2}{V_x^2} - \frac{\varepsilon \; k_x^2}{\varepsilon(1+\delta)}$$

$$(k^y)^2 = \left(\frac{\omega^2}{V_x^2} - k_x^2\right) + \delta k_x^2$$

So defining $k_{(0)} \equiv \omega/(c/n)$ (i.e. the normal thing)

$$k_x^2 + k_y^2 = k^2 = k_{(0)}^2 + \delta k_x^2 \implies k = k_{(0)} + \frac{1}{2}\delta \frac{k_x^2}{k_{(0)}}$$

So

$$\sin\theta_R = \frac{k^x}{k} = \frac{k^x}{\left(k_{(0)} + \frac{1}{2}\delta \frac{k_x^2}{k_{(0)}}\right)}$$

$$\simeq \frac{k^x}{k_{(0)}}\left(1 - \frac{\delta}{2}\frac{k_x^2}{k_{(0)}^2}\right)$$

$$\sin\theta_R = \sin\theta_{(0)}\left(1 - \frac{1}{2}\delta \sin^2\theta_{(0)}\right)$$

where $\sin\phi_{(0)}$ is the "normal/usual" refraction angle

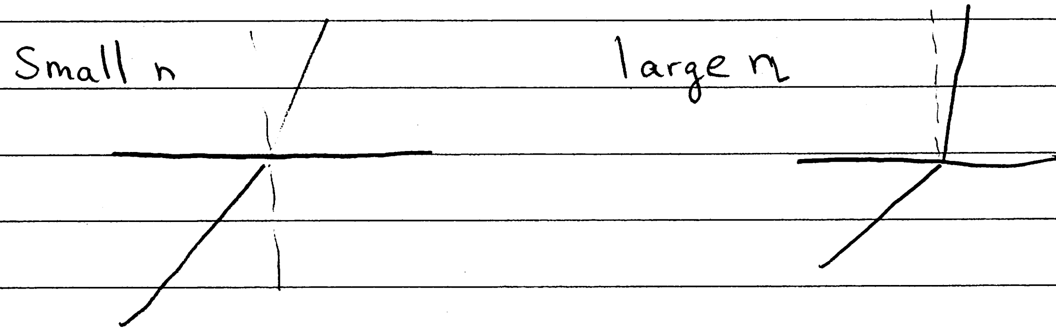
Snell's Law pg. 6

Using the result from part (a)

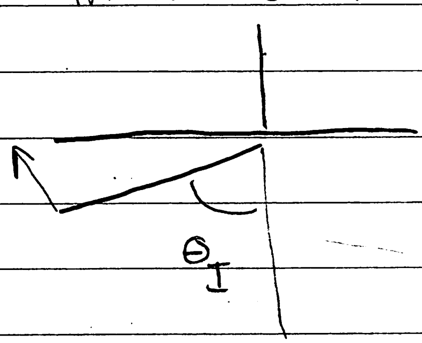
$$\boxed{\sin\theta_R = \frac{1}{n}\sin\theta_I\left(1 - \frac{\delta}{2}\frac{\sin^2\theta_I}{n^2}\right)}$$

d) The refracted angle is smaller, this make sense the index of refraction is larger in one direction

For a larger n in the isotropic case the refraction angle is smaller



The larger is the angle of incidence the more the transmitted light is polarized along the slow axis (larger index of refraction axis), i.e.



For larger θ_I the electric field is more polarized along y, so the index of refraction is larger than normal

This explains the larger shift at larger incidence, θ_I

e) If the electric field is in the z-direction, then it is aligned along a single axis of the crystal. we have simply:

$$k^2 = \frac{\omega^2}{V_z^2}$$

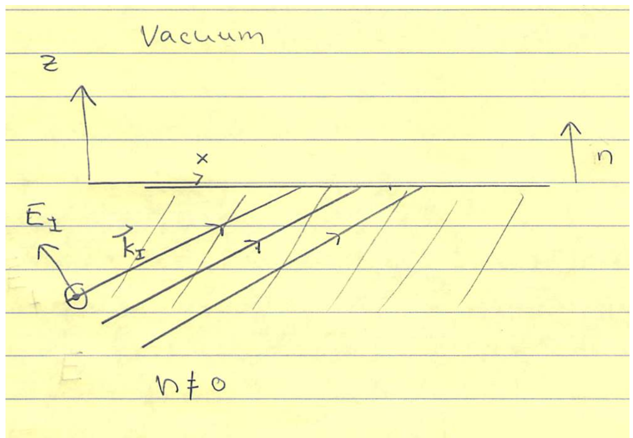
This ray then obeys the ordinary snell's law with the index being the index in the z-direction.

$$\sin\theta_2 = \frac{k^x}{k} = \frac{k^x}{\sqrt{\varepsilon_z}\,(\omega/c)} =$$

$$\sin\theta_2 = \frac{1}{n}\sin\theta_1 \qquad n \equiv \frac{1}{\sqrt{\varepsilon}}$$

Problem 4. Analysis of the Good-Hänchen effect

A "ribbon" beam[1] of in plane polarized radiation of wavelength λ is totally internally reflected at a plane boundary between a non-permeable (i.e. $\mu = 1$) dielectric media with index of refraction n and vacuum (see below). The critical angle for total internal reflection is θ_I^o, where $\sin\theta_I^o = 1/n$. First assume that the incident wave takes the form $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}_I e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ of a pure plane wave polarized in plane and study the transmitted and reflected waves.



(a) Starting from the Maxwell equations, show that for $z > 0$ (i.e. in vacuum) the electric field takes the form:

$$\boldsymbol{E}_2(x, z) = \boldsymbol{E}_2 e^{-\frac{\omega}{c}(\sqrt{n^2 \sin\theta_I^2 - 1})z} e^{i\frac{\omega n \sin\theta_I}{c}x} \tag{13}$$

(b) Starting from the Maxwell equations, show that for $\theta_I > \theta_I^0$ the ratio of the reflected amplitude to the incident amplitude is a pure phase

$$\frac{E_R}{E_I} = e^{i\phi(\theta_I, \theta_I^o)} \tag{14}$$

and determine the phase angle. Thus the reflection coefficient $R = |E_R/E_I|^2 = 1$ However, phase has consequences.

(c) Show that for a monochromatic (*i.e.* constant $\omega = ck$) ribbon beam of radiation in the z direction with a transverse electric field amplitude, $E(x)e^{ik_z z - i\omega t}$, where $E(x)$ is smooth and finite in the transverse extent (but many wavelengths broad), the lowest order approximation in terms of plane waves is

$$\boldsymbol{E}(x, z, t) = \boldsymbol{\epsilon} \int \frac{d\kappa}{(2\pi)} A(\kappa) e^{i\kappa x + ikz - i\omega t} \tag{15}$$

where $k = \omega/c$. Thus, the finite beam consists of a sum plane waves with a small range of angles of incidence, centered around the geometrical optics value.

[1]By a "ribbon" beam I mean a beam which has finite transverse extent in the direction perperndicular $\boldsymbol{k}_I$ lying in the x-z place as drawn above. But, the beam is infinite in extent in the y direction (coming out of page in the figure above) . Thus the incoming and outgoing "ribbion" beams form a kind of wedge.

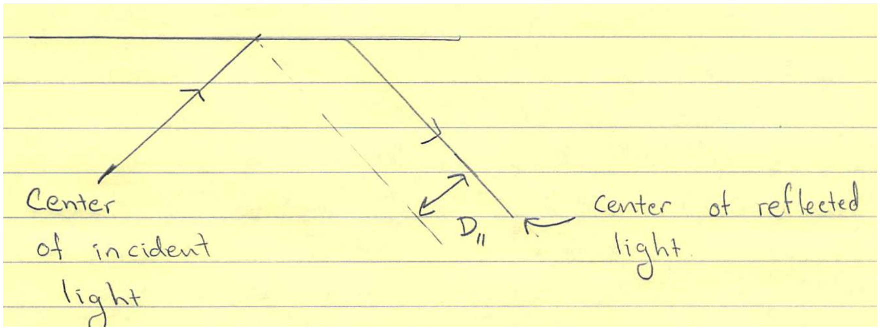
(d) Consider a reflected ribbon beam and show that for $\theta_I > \theta_I^o$ the electric field can be expressed approximately as

$$\boldsymbol{E}_R = \boldsymbol{\epsilon}_R E(x'' - \delta x) e^{i\boldsymbol{k}_R\cdot\boldsymbol{r} - i\omega t + i\phi(\theta_I, \theta_I^o)} \tag{16}$$

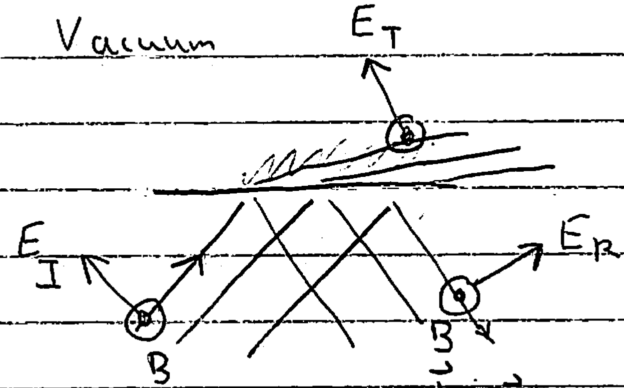
where $\boldsymbol{\epsilon}_R$ is a polarization vector, x'' is the coordinate perpendicular to the reflected wave vector $\boldsymbol{k}_R$, and the displacement $\delta x = -\frac{1}{k}\frac{d\phi}{d\theta_I}$ is determined by phase shift.

(e) Using the phase shift you computed, show that the lateral shift of the reflected in plane polarized beam is

$$D_\parallel = \frac{\lambda}{\pi}\frac{\sin\theta_I}{\sqrt{\sin^2\theta_I - \sin^2\theta_I^o}}\,\frac{\sin^2\theta_I^o}{\sin\theta_I^2 - \cos\theta_I^2\sin^2\theta_I^o} \tag{17}$$



Goos-Hänchen Effect



a) $$\vec{E}_1 = \vec{E}_I e^{i\vec{k}_I \cdot \vec{x}} + \vec{E}_R e^{i\vec{k}_R \cdot \vec{x}}$$

$$\vec{E}_2 = \vec{E}_T e^{i\vec{k}_T \cdot \vec{x}}$$

$$k_I^2 = \frac{\omega^2 n^2}{c^2} = k_R^2 \quad , \quad \vec{k}_T^2 = \frac{\omega^2}{c^2}$$ $\Rightarrow$ these come from Helmholtz

From the continuity of the phase

$$(\vec{k}_I \cdot \vec{x}) = (\vec{k}_R \cdot \vec{x})\Big|_{z=0} = (\vec{k}_T \cdot \vec{x})\Big|_{z=0}$$

Or

$$k_{Ix} = k_{Rx} = k_{Tx}$$

So

$$k_I \sin\theta = k_{Tx}$$

$$\frac{\omega n}{c} \sin\theta = k_{Tx}$$

And $$k_{Tx}^2 + k_{Tz}^2 = \frac{\omega^2}{c^2}$$

From $\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0$

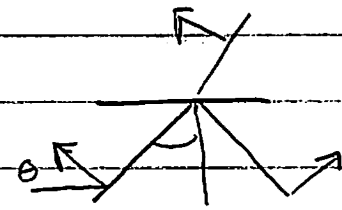
and $\frac{1}{Z} \hat{k} \times \vec{E} = \vec{H}$ where $Z = \sqrt{\frac{\mu}{\varepsilon}}$

We have with $\vec{H}$ polarized out of Plane:

$$\boxed{(H_I + H_R) - H_T = 0}$$

and from

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$



So

$$E_{Tx} - (-E_{Ix} + E_{Rx}) = 0$$

$$\boxed{E_{Tx} + E_I \cos\theta_I - E_R \cos\theta_I = 0}$$

Here E_{Tx} is defined by formal analogy

$$E_{Tx} = -E_t \cos\theta_T = -E_t \frac{k_z}{\sqrt{k_x^2 + k_y^2}} = -i E_t \frac{K_z}{\omega/c}$$

$$= \underbrace{-i\,(n^2 \sin^2\theta - 1)^{1/2}}_{\equiv \cos\theta_T} E_T$$

So the eqs to be solved are:

$$(E_I + E_R) - n E_T = 0 \qquad \text{we used } Z = \frac{1}{\sqrt{\varepsilon}} = \frac{1}{n}$$

and

$$-E_T \cos\theta_T + (E_I - E_R)\cos\theta_I = 0$$

So

$$n(E_I + E_R) = E_T$$

$$-n(E_I + E_R)\cos\theta_T + (E_I - E_R)\cos\theta_I = 0$$

$$\frac{E_I(\cos\theta_I - n\cos\theta_T)}{(\cos\theta_I + n\cos\theta_T)} = E_R$$

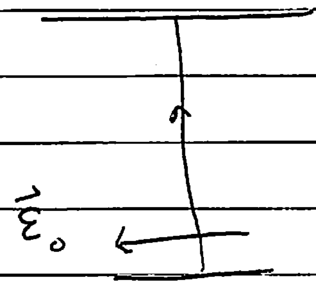
So since $\cos\theta_T$ is purely imaginary

$$\boxed{\frac{E_R}{E_I} = \frac{C_I - inC_T}{C_I + inC_T}} \qquad \boxed{\begin{array}{l} C_T = \sqrt{n^2\sin^2\theta - 1} \\ C_I = \cos\theta_I \end{array}}$$

So $$\frac{E_R}{E_I} = \frac{Re^{-i\alpha}}{Re^{i\alpha}} = \underbrace{e^{-2i\alpha}}_{e^{i\phi}} \quad \text{and} \quad \boxed{\phi = -2\tan^{-1}\left(\frac{nC_T}{C_I}\right)}$$

where $Re^{i\alpha} = C_I + inC_T$

c)



$\vec{k} = (\Delta k, k_0)$ ← exact $\vec{k}$

$\vec{k}_0 = (0, k_0)$ ← central k

$$E(x) = \int \frac{d\Delta k}{(2\pi)} A(\Delta k)\, e^{i\Delta k x}$$

The exact field in Fourier space: $\sqrt{(\Delta K)^2 + k_0^2} = \omega/c$

$$\vec{E}(\vec{k}, t) = \vec{\mathcal{E}}(k)\, E_0\, e^{i\Delta k x + ik_0 z - i\omega t} A(\Delta K)$$

Where $\vec{k} \cdot \vec{\mathcal{E}}(k) = 0$

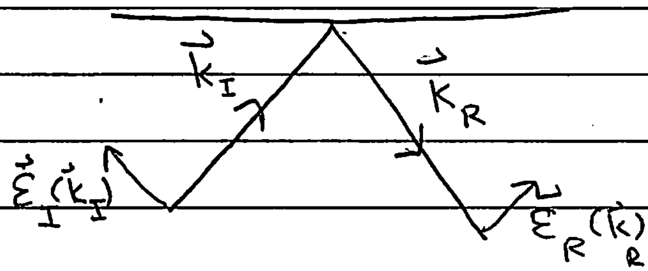
Using $k = \sqrt{k_0^2 + (\Delta k)^2} \simeq k_0$ and $\vec{k}_0 \cdot \hat{\mathcal{E}}_0 = 0$ $\vec{\mathcal{E}}(k) \simeq \vec{\mathcal{E}}_0$

We have:

exact ⟶ $\vec{E}(x, z, t) = \int \frac{d\Delta k}{2\pi} \vec{\mathcal{E}}(k)\, A(\Delta k)\, e^{i\Delta k x + ik_0 z - i\omega t}$

$$\vec{E}(x, z, t) \simeq \vec{\mathcal{E}}_0 \int \frac{d\Delta k}{2\pi} A(\Delta k)\, e^{i\Delta k x + ikz - i\omega t}$$

Now consider the reflection of the wave packet. We derived that:



$$E_R(k) = e^{i\phi(\theta_I)} \; E_I(k) \quad (\text{Amplitude})$$

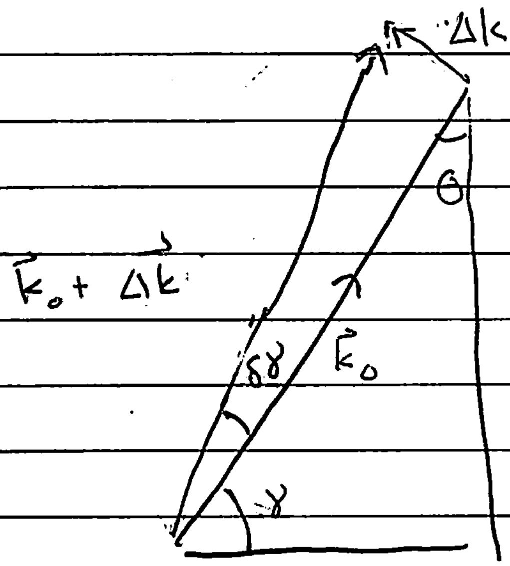
$$k_R = (k_{Ix}, -k_{Iz})$$

So

$$\vec{E}_R(\tilde{x}, \tilde{z}, t) = \int \vec{\mathcal{E}}_R(\vec{k}_R) \, e^{i\vec{k}_R \cdot \vec{x}} \, \overbrace{e^{i\phi(\theta_I)} \underbrace{A(\Delta k)}_{E_I(k)}}^{E_R(k)} \, \frac{d\Delta k}{2\pi}$$

Here $\phi(\theta_I)$ is a function of $\vec{k}_I$
From this picture, we see that that changing $\vec{k}_0 \to \vec{k} + \Delta k$ changes the angle:

$$\delta\gamma = + \frac{\Delta k}{k}$$



$$\delta\theta_I = -\delta\gamma = \boxed{-\frac{\Delta k}{k} = \delta\theta_I}$$

Thus adding ΔK decreases the angle of incidence

Upon reflection $\vec{k}_R$ is related to k_I

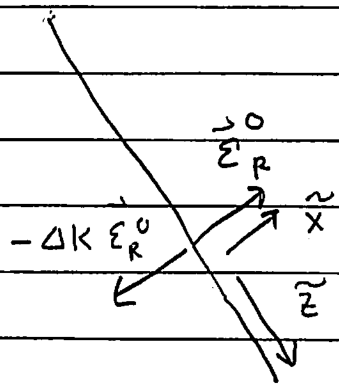
$$k_I = (\vec{k}_0 + \Delta\vec{k})_I = (k\sin(\theta_I + \delta\theta_I),\ k\cos(\theta_I + \delta\theta_I))$$

$$\vec{k}_R = (k\sin(\theta_I + \delta\theta_I),\ -k\cos(\theta_I + \delta\theta_I))$$

$$= k(\sin\theta_I, -\cos\theta_I) + k\,\delta\theta\,\underbrace{(\cos\theta_I, \sin\theta_I)}_{\vec{\varepsilon}^{\,0}_R}$$

$$= \vec{k}^0_R + \Delta k\,\vec{\varepsilon}^{\,0}_R$$

Thus Δk is conjugate to a shift in the $-\tilde{x}$ direction. Here $\tilde{x}$ and $\tilde{z}$ are the coordinates shown below.



The reflected wave is

$$E_R(\tilde{x}, \tilde{z}, t) = \vec{\varepsilon}^{\,0}_R \int \frac{d\Delta k}{2\pi}\, e^{-i\Delta k\,\tilde{x}}\, e^{ik\tilde{z} - i\omega t}\, e^{i\phi(\theta_I)}\, A(\Delta k)$$

So expanding

$$\phi(\theta_I) = \phi^0 + \frac{d\phi}{d\theta_I}\delta\theta_I$$

$$= \phi^0 - \frac{d\phi}{d\theta_I}\frac{\Delta K}{K}$$

So Fourier transforming

$$\vec{E}_R = \vec{\mathcal{E}}^0_R \int \frac{d\Delta K}{2\pi} A(\Delta K)\, e^{-i\Delta K\tilde{x} + ik_R\tilde{z}}\, e^{i\phi^0}\, e^{-i\frac{d\phi}{d\theta_I}\frac{\Delta k}{K}}$$

$$\vec{E}_R = \vec{\mathcal{E}}^0_R\, E(-(\tilde{x} - \delta\tilde{x}))\, e^{ik_R\tilde{z}}\, e^{i\phi^0}$$

So

$$\boxed{\delta\tilde{x} = -\frac{d\phi}{d\theta_I}\frac{1}{K}}$$

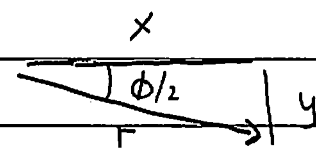
We also see that the wave form is inverted relative to the polarization vector

The Last part is algebra

$$\phi(\theta_I) = -2 \operatorname{atan} \frac{n C_T}{C_I}$$

$$-\tan \frac{\phi}{2}(\theta_I) = \frac{1}{S_o^2} \frac{\sqrt{\sin^2\theta - S_o^2}}{\sqrt{1-\sin^2\theta}} \equiv \frac{y}{x}$$

$$-\sec^2 \frac{\phi}{2} \, \frac{1}{2} \frac{d\phi}{d\theta_I} = \frac{d}{d\theta}\left(\frac{1}{S_o^2} \frac{\sqrt{s^2 - s_o^2}}{\sqrt{1-s^2}} \right)$$



$$\frac{d\phi}{d\theta_I} = -2\cos^2(\phi/2) \, \frac{d}{d\theta}\left(\frac{y}{x}\right)$$

So using mathematica:

$$-\frac{1}{K} \frac{d\phi}{d\theta_I} = +\frac{2}{K} \frac{\sin\theta_I}{(\sin^2\theta_I - S_o^2)^{1/2}} \frac{S_o^2}{(s^2 - c^2 S_o^2)}$$

$$= \frac{\lambda}{\pi} \frac{\sin\theta_I}{(\sin^2\theta_I - S_o^2)^{1/2}} \frac{S_o^2}{(\sin^2\theta - \cos^2\theta \, S_o^2)}$$

▪ Doing derivatives for Goos Hanchen.

In[130] := `yy = Sqrt[Sin[θ]^2 - S₀^2 ]`

Out[130]= $\sqrt{\mathrm{Sin}[\theta]^2 - S_o^2}$

In[131] := `xx = S₀^2 Sqrt[ 1 - Sin[θ ]^2 ]`

Out[131]= $\sqrt{1 - \mathrm{Sin}[\theta]^2}\; S_o^2$

In[132] := `rr = Sqrt[xx^2 + yy^2 ]`

Out[132]= $\sqrt{\mathrm{Sin}[\theta]^2 - S_o^2 + \left(1 - \mathrm{Sin}[\theta]^2\right) S_o^4}$

```
(* Compute -1*dphi/dthotaI *)
```

In[137] := `-Simplify[ -2 xx^2 / rr^2 D[yy / xx, θ] , Assumptions → Cos[θ] > 0]`

Out[137]= $$\frac{2\,\mathrm{Sin}[\theta]\; S_o^2}{\sqrt{\mathrm{Sin}[\theta]^2 - S_o^2}\;\left(\mathrm{Sin}[\theta]^2 - \mathrm{Cos}[\theta]^2 S_o^2\right)}$$

```
(* Alternate method we return to ER/EI = Exp[I phi]  or phi = -I Log [ ER/EI ] *)
```

In[128] := `-FullSimplify[D[-I Log[ (yy - I xx) / (yy + I xx)], θ], Assumptions → Cos[θ] > 0 ]`

Out[128]= $$\frac{2\,\mathrm{Sin}[\theta]\; S_o^2}{\sqrt{\mathrm{Sin}[\theta]^2 - S_o^2}\;\left(\mathrm{Sin}[\theta]^2 - \mathrm{Cos}[\theta]^2 S_o^2\right)}$$

Problem 5. Reflection of a Gaussian Wave Packet Off a Metal Surface:

In class we showed that the amplitude reflection coefficient from a good conductor ($\omega \ll \sigma$) for a plane wave of wavenumber $k = \omega/c$ is

$$\frac{H_R(k)}{H_I(k)} = 1 - \sqrt{\frac{2\mu\omega}{\sigma}}(1-i) \simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi(\omega)}\,, \tag{18}$$

where the phase is for $\omega \ll \sigma$:

$$\phi(\omega) \simeq \sqrt{\frac{2\mu\omega}{\sigma}}\,. \tag{19}$$

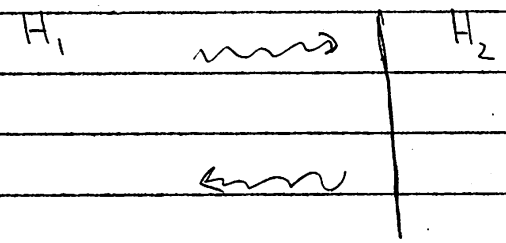
Consider a Gaussian wave packet with average wave number k_o centered at $z = -L$ at time $t = -L/c$ which travels towards a metal plane located at $z = 0$ and reflects. Show that the time at which the center of the packet returns to $z = -L$ is given by

$$t = \frac{L}{c} + \frac{\mu\delta_o}{2c} \tag{20}$$

where the time delay is due to the phase shift $d\phi(\omega_o)/d\omega$, and $\delta_o = \sqrt{2c/\sigma\mu k_o}$ is the skin depth.

Problem

- When analyzing the reflection of light off metal:



We showed that:

$$H_1 = H_I e^{ikz - i\omega t} + H_R e^{-ikz - i\omega t}$$

where

$$\frac{H_R}{H_I} = 1 - \sqrt{\frac{4\mu\omega}{\sigma}} \; \frac{1-i}{\sqrt{2}}$$

$$= 1 - \sqrt{\frac{2\mu\omega}{\sigma}} + i\sqrt{\frac{2\mu\omega}{\sigma}}$$

$$\simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi}$$

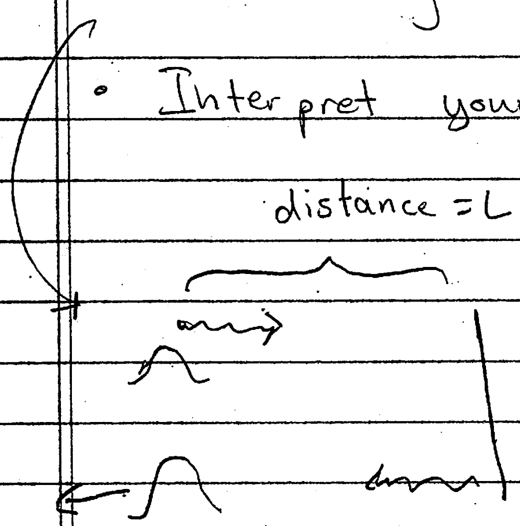
where $\tan\phi \simeq \sin\phi \simeq \phi = \sqrt{\frac{2\mu\omega}{\sigma}}$

Now study a wave packet propagating into the metal.

- Show that the phase is irrelevant for the reflection coeffecient

- But, show that the phase causes to a time delay between the naive (geometric optic) arrival time and actual arrival time of the reflected pulse. Compute the time delay.

- Interpret your result:



The time it takes before the pulse returns is

$$\Delta t = \frac{2L}{c} + \text{bit}$$

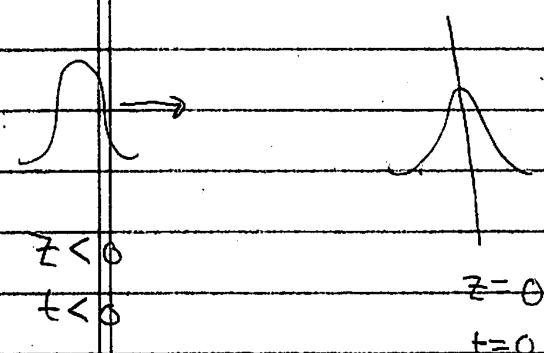
determine this.

Solution:

The incoming wave packet has

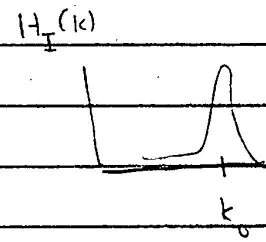
$$H_I(x,t) = \int \frac{dk}{2\pi}\, e^{ikz - i\omega t}\, H_I(k)$$

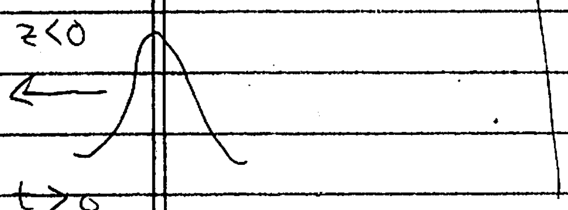
The phases are chosen so that the pulse hits the metal at $z=0$ and $t=0$



$$H_I^0 \equiv H_I(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, H_I(k)\, e^{ikx}$$

$= \text{gaussian}$





Then $H_R(x,t) = \int \frac{dk}{2\pi}\, e^{-ikz - ickt}\, H_I(k)\, r(k)\, e^{i\phi(k)}$

where

$$r(k) = 1 - \sqrt{\frac{2\mu\omega}{\sigma}} \qquad \phi(k) = \sqrt{\frac{2\mu\omega}{\sigma}}$$

Expanding the phase

$$\phi(k) = \phi(k_0) + \left.\frac{d\phi}{dk}\right|_{k_0}(k-k_0)$$

We have

$$H_R(z,t) = \int \frac{dk}{2\pi}\, e^{-ikz}\, e^{-ickt}\, H_I(k)\, r(k_0)\, e^{i\phi(k_0) + i\frac{d\phi}{dk}(k-k_0)}$$

And $e^{i\Phi} \equiv$ irrelevant phase

$$H_R(z,t) = r(k_0)\, \overbrace{e^{i\phi(k_0) - i\frac{d\phi}{dk}k_0}} \int \frac{dk}{(2\pi)}\, e^{ik\left(-z-ct+\frac{d\phi}{dk}\right)}\, H_I(k)$$

$$\simeq r(k_0)\, e^{i\Phi}\, H_I\left(-z-ct+\frac{d\phi}{dk}\right)$$

So the center of the wave packet reaches $z=-L$ when

$$-(-L) - ct + \left.\frac{d\phi}{dk}\right|_{k_0} = 0$$

$$\frac{L}{c} + \frac{1}{c}\left.\frac{d\phi}{dk}\right|_{k_0} = t \implies t = \frac{L}{c} + \frac{\mu\delta_0}{2c}$$

Substituting $\qquad \omega_0 \equiv ck_0$

$$\phi = \sqrt{\frac{2\mu ck}{\sigma}} \Rightarrow \frac{1}{2}\sqrt{\frac{2\mu}{\sigma ck}} = \frac{d\phi}{dk}\frac{1}{c} = \frac{\mu\delta_0}{2c} \qquad \delta_0 \equiv \sqrt{\frac{2c^2}{\sigma\mu\omega_0}}$$