Problem 1. Dipole from potentials to order $1/c^2$

This continues the "Dipole two ways" problem from the homework

(a) (Optional) Starting from the Maxwell equations derive/write down the equations for $(\phi, \boldsymbol{A})$ in the Lorenz gauge

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \boldsymbol{A} = 0 \tag{1}$$

and Coulomb gauges

$$\nabla \cdot \boldsymbol{A} = 0 \tag{2}$$

(b) Use your expressions to shown that to first order in $1/c$

$$\boldsymbol{A}_{\rm lrnz}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}(\boldsymbol{r}_0)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{3}$$

$$\boldsymbol{A}_{\rm coul}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}(\boldsymbol{r}_0)/c + \boldsymbol{j}_{\rm D}(\boldsymbol{r}_0)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{4}$$

Evaluate the Lorenz gauge integral (using the results of homework 2) yielding

$$\boldsymbol{A}_{\rm lrnz} = \frac{\dot{\boldsymbol{p}}}{4\pi r c} \tag{5}$$

(c) Show that the Coulomb gauge expression can be written

$$\boldsymbol{A}_{\rm coul}(\boldsymbol{r}) = \frac{\dot{\boldsymbol{p}}}{4\pi r c} - \nabla \frac{\partial}{c\partial t}\left[\int d^3\boldsymbol{r}_0\, \varphi(\boldsymbol{r}_0)\, \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}\right] \tag{6}$$

At home (or in class if you have time) show that[1]

$$\left[\int d^3\boldsymbol{r}_0\, \varphi(\boldsymbol{r}_0)\, \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}\right] = \frac{\boldsymbol{p}\cdot\boldsymbol{n}}{8\pi} \tag{9}$$

where $\boldsymbol{n} \equiv \hat{\boldsymbol{r}}$, to show that

$$\boldsymbol{A}_{\rm coul}(\boldsymbol{r}) = \frac{\dot{\boldsymbol{p}}}{4\pi r c} - \nabla \frac{\partial}{c\partial t}\left(\frac{\boldsymbol{n}\cdot\boldsymbol{p}}{8\pi}\right) \tag{10a}$$

$$= \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\dot{\boldsymbol{p}}) + \dot{\boldsymbol{p}}}{8\pi r c} \tag{10b}$$

As a by product you should find

$$\partial_i n_j = \frac{\delta_{ij} - n_i n_j}{r} \tag{11}$$

which will be relatively useful going forward.

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[1] Use the "Coulomb Identity"

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} = \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi) Y_{\ell m}(\theta_0,\phi_0) \tag{7}$$

and the trick in lecture with

$$\varphi(\boldsymbol{r}_0) = \frac{p\cos(\theta_0)}{4\pi r_0^2} \tag{8}$$

(d) Without calculation explain why the magnetic field $\boldsymbol{B}^{(1)}$ from Eq. (10) and Eq. (5) must agree. Which gauge is easier for the magnetic field?

(e) Starting from the equations written down in (a), determine the correction to order $1/c^2$ to φ and in the Lorenz and Coulomb gauges. You should find :

$$\varphi^{(2)}_{\rm lrnz} = -\frac{\boldsymbol{n}\cdot\ddot{\boldsymbol{p}}}{8\pi c^2} \tag{12}$$

Relate the two results for $(\varphi, \boldsymbol{A})$ via a gauge transformation.

(f) Determine the electric field to second order in $1/c$ using the Lorenz and Coulomb gauges. Notice how particularly simple the Coulomb gauge is for this purpose. You should find (in either gauge)

$$\boldsymbol{E}^{(2)} = -\frac{\boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{p}}) + \ddot{\boldsymbol{p}}}{8\pi r c^2} \tag{13}$$

(g) At what radius does $\boldsymbol{E}^{(2)}$ become comparable to $\boldsymbol{E}^{(0)}$

Solution

(a) See lecture. In the Lorentz gauge

$$-\Box\varphi = \rho \tag{14}$$

$$-\Box \boldsymbol{A} = \frac{\boldsymbol{j}}{c} \tag{15}$$

In the Coulomb gauge

$$-\nabla^2\varphi = \rho \tag{16}$$

$$-\Box \boldsymbol{A} = \frac{\boldsymbol{j}}{c} + \frac{\boldsymbol{j}_D}{c} \tag{17}$$

where the displacement current is

$$\frac{\boldsymbol{j}_D}{c} = -\frac{1}{c}\partial_t \nabla\varphi \tag{18}$$

(b) In the Coulomb gauge

$$(\frac{1}{c^2}\partial_t^2 - \nabla^2)\boldsymbol{A} = \boldsymbol{j}/c + \frac{1}{c}\partial_t(-\nabla\varphi) \tag{19}$$

To first order in $1/c$ we may neglect the second derivative ∂_t^2. We may also replace φ with its zero order solution

$$-\nabla^2\boldsymbol{A} = \boldsymbol{j}/c + \frac{1}{c}\partial_t(-\nabla\varphi^{(0)}) \tag{20}$$

The solution to this equation is

$$\boldsymbol{A}(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}(t,\boldsymbol{r}_0)/c + \boldsymbol{j}_D(t,\boldsymbol{r}_0)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{21}$$

where the displacement current is

$$\boldsymbol{j}_D(t,\boldsymbol{r}_0) = \partial_t\boldsymbol{E}^{(0)} = -\partial_t(\nabla\varphi^{(0)}) \tag{22}$$

In the Lorenz Gauge

$$(\frac{1}{c^2}\partial_t^2 - \nabla^2)\boldsymbol{A} = \boldsymbol{j}/c \tag{23}$$

and to first order in $1/c$

$$\boldsymbol{A}(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}(t,\boldsymbol{r}_0)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}\,. \tag{24}$$

In the dipole approximation from the previous problem

$$\boldsymbol{j} = \dot{\boldsymbol{p}}(t)\delta^3(\boldsymbol{r}-\boldsymbol{r}')\,, \tag{25}$$

where $\boldsymbol{r}'$ is the position of the dipole, i.e. $\boldsymbol{r}'=0$ in this problem

So

$$\boldsymbol{A}_{\rm lrnz}(t,\boldsymbol{r}) = \frac{\dot{\boldsymbol{p}}}{4\pi rc} \tag{26}$$

(c) In the Coulomb gauge we have

$$\boldsymbol{A}_{\text{coul}} = \boldsymbol{A}_{\text{lrnz}} + \int d^3\boldsymbol{r}_0 \frac{\boldsymbol{j}_D(t,\boldsymbol{r}_0)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{27}$$

leading to

$$\boldsymbol{A}_{\text{coul}} = \boldsymbol{A}_{\text{lrnz}} - \nabla \frac{1}{c}\partial_t \underbrace{\int d^3\boldsymbol{r}_0 \frac{\varphi^{(0)}(t,\boldsymbol{r}_0)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}}_{\text{Integral}} \tag{28}$$

Substitute

$$\varphi(t,\boldsymbol{r}_0) = \frac{p(t)\cos(\theta_0)}{4\pi r_0^2} \tag{29}$$

into the underlined integral, use the expansion

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} = \sum_{\ell m} \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)Y_{\ell m}(\theta_0,\phi_0), \, , \tag{30}$$

and perform the integrals over angles θ_0, ϕ_0. The angular integrals are simplified by noting that

$$\cos(\theta_0) = AY_{10}(\theta_0,\phi_0) \tag{31}$$

with A a constant, and thus only the $\ell = 1$ and $m = 0$ term survives the integration, yielding $AY_{10}(\theta,\phi) = \cos\theta$. The radial integration gives

$$\begin{aligned}
\int d^3\boldsymbol{r}_0 \frac{\varphi^{(0)}(t,\boldsymbol{r}_0)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} &= p(t)\cos(\theta)\int_0^\infty \frac{r_0^2 dr_0}{4\pi r_0^2}\frac{1}{3}\frac{r_<}{r_>^2} && \text{(32a)}\\
&= \frac{p(t)\cos(\theta)}{4\pi}\left[\int_0^r dr_0 \frac{r_0}{3r^2} + \int_r^\infty dr_0 \frac{r}{3r_0^2}\right] && \text{(32b)}\\
&= \frac{p(t)\cos(\theta)}{4\pi}\left[\frac{1}{6}+\frac{1}{3}\right] && \text{(32c)}\\
&= \frac{p(t)\cos(\theta)}{8\pi} && \text{(32d)}\\
&= \frac{\boldsymbol{p}(t)\cdot\boldsymbol{n}}{8\pi} && \text{(32e)}
\end{aligned}$$

where $\boldsymbol{n} = \hat{\boldsymbol{r}}$.

Then note the derivative

$$\partial_i n_j = \frac{\delta_{ij} - n_i n_j}{r} \tag{33}$$

so the i-th component of the gradient is

$$[\nabla(\boldsymbol{n}\cdot\dot{\boldsymbol{p}})]_i = \frac{\dot{p}_i - (\dot{\boldsymbol{p}}\cdot\boldsymbol{n})n_i}{r} \tag{34}$$

When combined with $\dot{\boldsymbol{p}}/(4\pi rc)$ from $\boldsymbol{A}_{\text{lrnz}}$ in Eq. (28), we find the quoted result in Eq. (10)

(d) The two expressions for $\boldsymbol{A}$ differ by a gradient, and the curl of a gradient is zero. So $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is the same for either form of $\boldsymbol{A}$.

(e) In the Lorenz gauge we should solve

$$(\frac{1}{c^2}\partial_t^2 - \nabla^2)\varphi_{\rm lrnz} = \rho \tag{35}$$

Setting up an expansion through second order in $1/c$

$$\varphi = \varphi^{(0)} + \varphi^{(2)} \tag{36}$$

At zeroth order we have the Laplace equation

$$-\nabla^2\varphi^{(0)}_{\rm lrnz} = \rho \tag{37}$$

while at second (and higher order) the zeroth order solution acts like a source for $\varphi^{(2)}$

$$-\nabla^2\varphi^{(2)}_{\rm lrnz} = -\frac{1}{c^2}\ddot{\varphi}^{(0)}_{\rm lrnz}\,. \tag{38}$$

The zeroth order solution is simply the potential due to a dipole

$$\varphi^{(0)}_{\rm lrnz}(t, \boldsymbol{r}_0) = \frac{\boldsymbol{p}(t)\cdot\boldsymbol{n}_0}{4\pi r_0^2} \tag{39}$$

where $\boldsymbol{n}_0 = \hat{\boldsymbol{r}}_0$. Then the scalar potential at second order (i.e. the solution to Eq. (38)) can be written down immediately by analogy to the Coulomb law

$$\varphi^{(2)}_{\rm lrnz}(t, \boldsymbol{r}) = -\int d^3\boldsymbol{r}_0\, \frac{\ddot{\boldsymbol{p}}(t)\cdot\boldsymbol{n}_0}{c^2(4\pi r_0^2)}\, \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{40}$$

The integral is the same as before (see Eq. (32)) yielding

$$\varphi^{(2)}_{\rm lrnz}(t, \boldsymbol{r}) = -\frac{\ddot{\boldsymbol{p}}\cdot\boldsymbol{n}}{8\pi c^2} \tag{41}$$

Of course the integral should be the same – we are solving the same problem. Changing gauges merely shuffles the problem around. (This is prosaically referred to as conservation of shit.)

Thus the full result (through $1/c^2$) for the Lorenz gauges is

$$\varphi_{\rm lrnz}(t, \boldsymbol{r}) = \frac{\boldsymbol{p}(t)\cdot\boldsymbol{n}}{4\pi r^2} - \frac{\ddot{\boldsymbol{p}}\cdot\boldsymbol{n}}{8\pi c^2} \tag{42a}$$

$$\boldsymbol{A}_{\rm lrnz}(t, \boldsymbol{r}) = \frac{\dot{\boldsymbol{p}}}{4\pi c r} \tag{42b}$$

In the Coulomb gauge the zeroth order potential (the potential from a dipole) is exact to all orders, but the vector potential is more complicated as we have seen:

$$\varphi_{\rm coul}(t, \boldsymbol{r}) = \frac{\boldsymbol{p}(t)\cdot\boldsymbol{n}}{4\pi r^2} \tag{43a}$$

$$\boldsymbol{A}_{\rm coul}(t, \boldsymbol{r}) = \frac{\dot{\boldsymbol{p}}}{4\pi r c} - \nabla\frac{(\boldsymbol{n}\cdot\dot{\boldsymbol{p}})}{8\pi c} \tag{43b}$$

The two gauges are related by a gauge transformation

$$\Lambda(t, \boldsymbol{r}) = \frac{\dot{\boldsymbol{p}} \cdot \boldsymbol{n}}{8\pi c} \tag{44}$$

where

$$\varphi_{\rm lrnz}(t, \boldsymbol{r}) = \varphi_{\rm coul} - \frac{1}{c}\partial_t \Lambda(t, r) \tag{45}$$

$$\boldsymbol{A}_{\rm lrnz}(t, \boldsymbol{r}) = \boldsymbol{A}_{\rm coul} + \nabla\Lambda \tag{46}$$

(f) To compute the electric field to quadratic order in the Coulomb gauge

$$\boldsymbol{E} = -\nabla\varphi^{(0)} - \frac{1}{c}\partial_t \boldsymbol{A} \tag{47}$$

$$= \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(2)} \tag{48}$$

The first term arises at zeroth order and is the usual

$$\boldsymbol{E}^{(0)} = -\nabla(\frac{\boldsymbol{p} \cdot \boldsymbol{n}}{4\pi r^2}) = \frac{3\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{p}) - \boldsymbol{p}}{4\pi r^3} \tag{49}$$

while the second term arises from the coulomb gauge vector potential at first order

$$\boldsymbol{E}^{(2)} = -\frac{1}{c}\partial_t \boldsymbol{A}^{(1)} = -\frac{1}{c}\partial_t \left[\frac{\boldsymbol{n}(\boldsymbol{n} \cdot \dot{\boldsymbol{p}}) + \dot{\boldsymbol{p}}}{8\pi r c}\right] \tag{50}$$

which gives the quoted result. In the Lorenz gauge

$$\boldsymbol{E} = -\nabla\varphi^{(0)}_{\rm lrnz} - \nabla\varphi^{(2)}_{\rm lrnz} - \frac{1}{c}\partial_t \partial \boldsymbol{A}^{(1)} \tag{51}$$

and

$$\boldsymbol{E}^{(2)} = -\nabla\varphi^{(2)}_{\rm lrnz} - \frac{\ddot{\boldsymbol{p}}}{4\pi r c^2} \tag{52}$$

$$= -\frac{\boldsymbol{n}(\boldsymbol{n} \cdot \ddot{\boldsymbol{p}}) + \ddot{\boldsymbol{p}}}{8\pi r c^2} \tag{53}$$

(g) Comparison

$$E^{(0)} \sim \frac{p}{r^3} \tag{54}$$

while

$$E^{(2)} \sim \frac{\ddot{p}}{c^2 r} \sim \frac{\omega^2 p}{c^2 r} \tag{55}$$

Thus the two are equal when

$$r \sim \frac{c}{\omega} \tag{56}$$