1 Short problem on displacement current

Consider a charging circular capacitor of radius a and negligble separation $d \ll a$. Charges are fed to the bottom plate and removed from the top plate by a long straight wire running up the z axis interrupted only by the spacing bettween the plates. The charge on the bottom plate as a function of time

$$Q(t) = I_0\, t\,, \tag{1}$$

while the charge on the top plate has $Q(t) = -I_0 t$.

(a) Determine the magnetic field outside the parallel plate capacitor for $|z| > d/2$.

(b) Determine the magnetic field inside the parallel plate capcitor, $|z| < d/2$. Neglect any fringing.

(c) Now consider the current on the surface of the parallel plate capacitor

(i) Determine this current from the boundary conditions of magneto-statics.

(ii) Assume the charge per area $\sigma(t)$ is not a function of spatial position and increases in linearly in time. The charge density in the plates satisfies the continuity equation

$$\partial_t \sigma + \nabla \cdot \boldsymbol{K} = 0\,, \tag{2}$$

except at the center of the plates where the charge is injected by the external wires. Take the current $\boldsymbol{K} = K(\rho)\,\hat{\boldsymbol{\rho}}$ and solve for $K(\rho)$ from Eq. (2). The result should agree with (i).

For a vector in cylindrical coordinates the divergence is

$$\nabla \cdot \boldsymbol{V} = \frac{1}{\rho}\frac{\partial(\rho V^\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(V^\phi)}{\partial \phi} + \frac{\partial V^z}{\partial z} \tag{3}$$

(iii) Compute and interpret the divergence of $\boldsymbol{K}$. (Hint: the result has a divergent piece and δ-function piece. Interpret the δ-fcn piece and the regular piece.)

Remark: One could be tempted (based on part (a)) to think that it is the electric field that "makes" the magnetic field. However, as the displacement current is curl free it does not contribute to the magnetic field in the Biot-Savat law (see homework). In the Biot-Savat Law it is the surface current $\boldsymbol{K}$ which produces the magnetic field. The continuity condition Eq. (2) and the boundary conditions of electrodynamics guaranty that these two approaches will yield consistent results.

(a) The magnetic field from a wire is from Ampere's Law

$$B_\phi = \frac{(I_0/c)}{2\pi\rho} \tag{4}$$

(b) Inside we use Ampere's Law with the displacement current

$$\oint \boldsymbol{B}\cdot d\boldsymbol{\ell} = \frac{1}{c}\partial_t \int \boldsymbol{E}\cdot d\boldsymbol{a} \tag{5}$$

Yielding

$$B_\phi 2\pi\rho = \frac{1}{c}\partial_t \frac{Q(t)}{\pi a^2}\pi\rho^2 \tag{6}$$

Or

$$B_\phi = \frac{(I_0/c)\rho}{2\pi a^2} \tag{7}$$

(c) (i) The boundary conditions are

$$\boldsymbol{n}\times(\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}}{c} \tag{8}$$

Taking $\boldsymbol{n}$ to be $\hat{\mathbf{z}}$ we have $\boldsymbol{n}\times\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}$, and thus

$$\left(\frac{I_0\rho}{2\pi a^2} - \frac{I_0}{2\pi\rho}\right)(\boldsymbol{n}\times\hat{\boldsymbol{\phi}}) = K(\rho)\hat{\boldsymbol{\rho}} \tag{9}$$

So

$$K(\rho) = -\frac{I_0\rho}{2\pi a^2} + \frac{I_0}{2\pi\rho} \tag{10}$$

(ii) Using continuity we have

$$\partial_t\sigma = \frac{I_0}{\pi a^2} \tag{11}$$

So we must solve

$$\frac{1}{\rho}\partial_\rho(\rho K) = -\frac{I_0}{\pi a^2} \tag{12}$$

Solving this equation via integration

$$K = -\frac{I_0\rho}{2\pi a^2} + \frac{C}{\rho} \tag{13}$$

The constant of integration is adjusted so that at $\rho = 0$ so that flux of current emanating from the origin is I_0:

$$\lim_{\rho\to 0}\oint_{\text{small circle}} \boldsymbol{K}\cdot\boldsymbol{n}d\ell = \lim_{\rho\to 0} K 2\pi\rho = I_0 \tag{14}$$

This fixes $C = I_0/2\pi$ and

$$K = -\frac{I_0\rho}{2\pi a^2} + \frac{I_0}{2\pi\rho} \tag{15}$$

in agreement with above.

(iii) The divergence is

$$\nabla \cdot \boldsymbol{K} = \frac{-I_0}{\pi a^2} + I_0 \delta^2(\boldsymbol{\rho}) \tag{16}$$

The delta function reflects the fact that we are feeding the current to the plate by the lead wire.