Maxwell Equations + Induction + Energy in Mag fields

$$\nabla \cdot E = \rho_{mat} + \rho_{ext}$$

$$\nabla \times B = \frac{j_{mat}}{c} + \frac{j_{ext}}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

- Then for the material current we write

$$\frac{j_{mat}}{c} = \cancel{\frac{\sigma}{c}\vec{E}}^{\,0 \text{ for insulator}} + \frac{1}{c}\partial_t \vec{P} + \nabla \times \vec{m}$$

Then with continuity, $\rho_{mat} = -\nabla \cdot P$, $D \equiv E + P$, $H = \vec{B} - \vec{m}$, find

Cool!

$$\boxed{\begin{aligned} \nabla \cdot D &= \rho_{ext} \\ \nabla \times H &= \frac{j_{ext}}{c} + \frac{1}{c}\partial_t \overbrace{\vec{D}}^{\vec{E}+\vec{P}} \\ \nabla \cdot B &= 0 \\ -\nabla \times E &= \frac{1}{c}\partial_t \vec{B} \end{aligned}}$$

← maxwell eqs in simple matter

Last Time pg. 2 + Induction pg. 1

Then we expand in powers of c

- Electrostatics

$$\nabla \cdot D^{(0)} = \rho_{ext}$$

$$\nabla \times E^{(0)} = 0$$

- Magnetostatics:

$$\nabla \times H^{(1)} = j_{ext}/c \quad + \frac{1}{c}\partial_t D^{(0)}$$

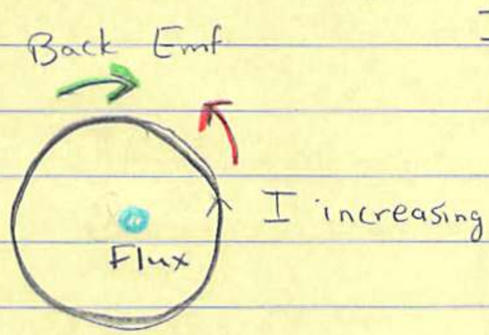
$$\nabla \cdot B^{(1)} = 0$$

- Induced Electric fields / Back Emf

$$\nabla \cdot D^{(2)} = 0$$

$$-\nabla \times E^{(2)} = \frac{1}{c}\partial_t B^{(1)} \qquad \text{could call } E^{(2)} = E^{ind}$$

Want to compute the energy stored in magnetic fields.



Imagine slowly increasing the current changing the current makes a changing magnetic field inducing a Back Emf. The work the Battery does to increase the current is the energy stored in the fields.

- Take almost magneto statics with $D^{(0)} = 0$ for simplicity:

$$\nabla \times H = j$$
$$\nabla \cdot B = 0$$

And

$$-\nabla \times E^{ind} = \frac{1}{c}\partial_t B$$

- Then the work by battery is $\delta\vec{E}_{batt} = -\delta\vec{E}_{ind}$

$$\frac{\delta U}{\delta t} = \frac{\delta W_{batt}}{\delta t} = -\int \vec{j} \cdot \delta E^{ind} / \delta t$$

$\vec{F} \cdot \vec{v}$

$$= -\int (\nabla \times H)\, c\, \delta E^{ind}$$

by parts:

$\nabla \cdot (H \times E) = (\nabla \times \vec{H}) \cdot E - H \cdot \nabla \times \vec{E}$

$$= -\int_V \vec{H} \cdot c\nabla \times \delta E^{ind}$$

$$\frac{\delta U}{\delta t} = \int_V \vec{H} \cdot \frac{\delta \vec{B}}{\delta t}$$

$$\boxed{\delta U = \int_V \vec{H} \cdot \delta \vec{B}}$$

- Then for linear media $\delta B = \mu \, \delta H$

$$U = \frac{1}{2}\int_V \frac{H^2}{\mu} = \boxed{\frac{1}{2}\int_V \vec{H}\cdot\vec{B}\, d^3x = U}$$

- These equations are often expressed in terms of $\vec{J}$ and $\vec{A}$ rather than $\vec{B}$

Indeed,

$$\delta U = \int_V \vec{H}\cdot\delta\vec{B} \qquad\qquad \delta E^{ind} = -\frac{1}{c}\partial_t \delta\vec{A} - \cancel{\vec{\nabla}\varphi}^{\,0}$$

$$\delta U = \int_V \vec{H}\cdot\vec{\nabla}\times\delta\vec{A}$$

By parts (no minus) because cross prod

$$\delta U = \int_V \underbrace{\nabla\times H}_{\vec{J}/c}\cdot\delta\vec{A}$$

$$\boxed{\delta U_B = \int_V \frac{\vec{J}}{c}\cdot\delta\vec{A}}$$

- For Linear media $\delta\vec{A} \propto \mu\,\delta\vec{J}$

$$\boxed{U = \frac{1}{2}\int \frac{\vec{J}}{c}\cdot\vec{A}}$$