Wires pg. 1

Inductance in Wires

① $$U_B = \frac{1}{2}\int_V \frac{\vec{j}}{c}\cdot\vec{A}\, d^3x = \frac{1}{2}\int_V \vec{H}\cdot\vec{B}$$

- U_B is a property of state

② $$\delta U_B = \int_V \frac{\vec{j}}{c}\cdot\delta\vec{A}$$

- For a set of wires: $\vec{j}\, d^3x = I\, d\vec{l}$

Then find

summed over a = loops

① $$U = \frac{1}{2}\frac{I_a}{c}\Phi_a \qquad \Phi_a = \oint_{a\text{-th loop}} \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{A}$$

② $$\delta U = \frac{I_a}{c}\,\delta\Phi_a$$ = flux through a-th loop

Note that, $$\vec{A}(x) = \mu\int_V \frac{j/c\,(x_0)}{4\pi|x-x_0|}$$

(☆) $$U_B = \frac{\mu}{2}\int d^3x\, d^3x_0\, \frac{\vec{j}(x)/c\cdot\vec{j}(x_0)/c}{4\pi|x-\vec{x}_0|}$$

Symmetric under interchange of $\vec{x}$ & $\vec{x}_0$

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- So for a set of wires

$$U_B = \frac{1}{2} I_a M_{ab} I_b$$

inductance matrix

M_{11} is the self inductance of the first loop

M_{12} is the mutual inductance between the 1st + 2nd loops.
($M_{12} = M_{21}$ since Eq ✱ is symmetric.)

- Then since $U_B = \frac{1}{2} \frac{I_a}{c} \Phi_a$

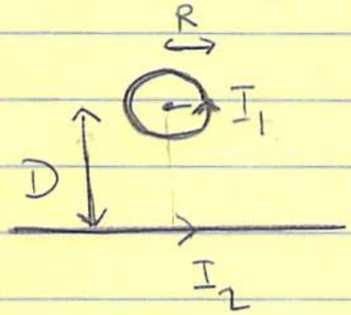
$$\frac{\Phi_a}{c} = M_{ab} I_b$$

back emf in a

And for any circuit $\mathcal{E}_a = -\frac{1}{c} \partial_t \Phi_B = -M_{ab} \frac{dI_b}{dt}$

Problem on Mutual Inductance & Force

- Compute the mutual inductance of a ring and a long straight wire



The force between the wire and ring is attractive. If currents are parallel they attract, if they are anti-parallel they repel (i.e. opposite to charges). Here the bottom end of the ring is closer to the wire and is attractive (the currents are parallel), while the upper end of the ring is farther away experiencing a weaker repulsive force (the currents are anti-parallel).

- Solution

current in wire one

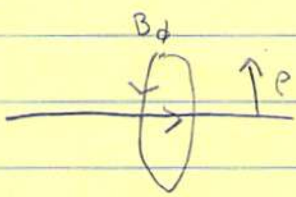
field from wire 2 at wire 1

$$U_{12} = \int_V \vec{J}_1 \cdot \vec{A}_2$$

$$= \frac{I_1}{c} \int \vec{A}_2 \cdot d\vec{l}_1 = \frac{I_1}{c} \int \vec{B}_2 \cdot d\vec{a}_1 = \frac{I_1}{c}\Phi_{21}$$

- Then the field from the wire is

$$\vec{B}_2 = \frac{I_2/c}{2\pi\rho}\hat{\phi}$$



So we need to integrate this field from the wire over the area of the ring

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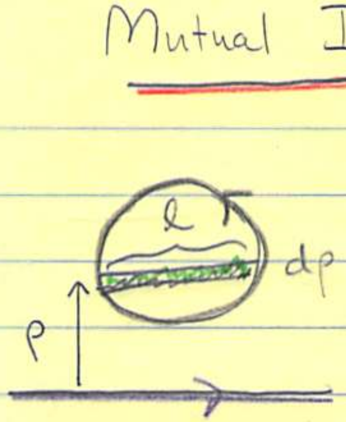

points out — given by circulation of I_1

$$\vec{B}_2 \cdot \hat{n}\, d\vec{a} = \frac{I_2/c}{2\pi\rho}\; \overbrace{2\left(R^2-(\rho-D)^2\right)^{1/2}}^{\ell \text{ (see picture)}}\, d\rho$$

points out

- We have

$$U_{12} = \int_{D-R}^{D+R} d\rho \; \frac{I_1 I_2}{c^2}\, \frac{2\left(R^2-(\rho-D)^2\right)^{1/2}}{2\pi\rho}$$

$$U_{12} = \frac{I_1 I_2}{c^2}\left[D - \sqrt{D^2-R^2}\right]$$

So $\quad M_{12} = \dfrac{1}{c^2}\left(D - \sqrt{D^2-R^2}\right)$

- Then we might want to compute the force between the ring and the wire. To do this we ask about the change in U_B, as the distance between the ring and the wire is changed with currents fixed:

$$\delta U_B = \underbrace{I_a\, \delta\Phi_a/c}_{\delta W_{batt}} + \delta W_{mech}$$

δW_{mech}: work done on system by mechanical forces $= -\vec{F}_{ring} \cdot \delta \vec{D}$

δU_B: Change in energy Stored in fields

δW_{batt}: work done by battery to keep current fixed.

$\vec{F}_{ring}$: force on ring

$\vec{F}_{ring} = -\vec{F}_{\text{applied mechanically}}$

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$$\delta U = \frac{1}{2} I_a \, \delta m_{ab} \, I_b$$

We are keeping the currents fixed here

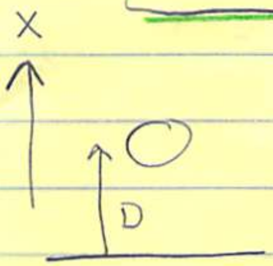
$$\frac{I_a}{c} \delta \Phi_a = I_a \, \delta m_{ab} \, I_b \leftarrow \delta W_{batt}$$

So

$$\delta U_B - \frac{I_a}{c} \delta \Phi_a = -\frac{1}{2} I_a \, \delta m_{ab} \, I_b = -\vec{F}_{ring} \cdot \delta \vec{D}$$

So

$$\boxed{F^x_{ring} = + \frac{\delta m_{ab}}{\delta D} \frac{1}{2} I_a I_b} = \frac{I_1 I_2}{c^2} \left(1 - \frac{D}{\sqrt{D^2 - R^2}} \right)$$



$$= -\frac{I_1 I_2}{c^2} \left(\frac{D}{\sqrt{D^2 - R^2}} - 1 \right)$$

indicates an attractive force
i.e. force in negative x-direction