Gauge Potentials

- Talked about inductance

$$\nabla \cdot D = \rho$$

$$\nabla \times H = j^{ext}/c + \frac{1}{c}\partial_t \vec{D}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

Then

0:
$$\nabla \cdot D^{(0)} = \rho$$
$$\nabla \times E^{(0)} = 0$$

1st
$$\nabla \times H^{(1)} = j^{ext}/c + \frac{1}{c}\partial_t \vec{D}^{(0)}$$
$$\nabla \cdot B^{(1)} = 0$$

2nd
$$-\nabla \times E^{(2)} = \frac{1}{c}\partial_t B^{(1)}$$

So concluded:

$$\delta U_B = \int_V H \cdot \delta B = \int_V \frac{\vec{j}}{c}\, \delta \vec{A}$$

Integrate
$\delta B \propto \delta H$

$$U_B = \frac{1}{2}\int_V H \cdot B = \frac{1}{2}\int_V \vec{j} \cdot \vec{A}$$

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Can also express the expansion in $1/c$ in terms of the gauge potentials $(\phi, \vec{A})$

① $\nabla \cdot E = \rho$

② $\nabla \times B = \vec{j}/c + \frac{1}{c}\partial_t \vec{E}$

③ $\nabla \cdot B = 0$

④ $-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$

- So then

$$\vec{B} = \nabla \times \vec{A} \qquad \text{from ③}$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi \qquad \text{from ④}$$

- Then from ①

$$\nabla\left(-\frac{1}{c}\partial_t \vec{A} - \nabla\varphi\right) = \rho$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi - \frac{1}{c}\frac{\partial}{\partial t}\left(+\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\vec{A}\right) = \rho$$

$$\boxed{-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right) = \rho}$$

same as ①

$$\Box \equiv -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 = \text{d'Alembertian}$$

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And from ②

$$\underbrace{\nabla \times (\nabla \times \vec{A})}_{\vec{\nabla}(\nabla\cdot\vec{A}) - \nabla^2 \vec{A}} = \vec{j}/c + \frac{1}{c}\partial_t\left(-\frac{1}{c}\partial_t \vec{A} - \nabla \varphi\right)$$

Then:

$$-\left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right)\vec{A} + \nabla\cdot\left(\frac{1}{c}\partial_t \varphi + \nabla\cdot\vec{A}\right) = j/c$$

i.e

$$\boxed{-\Box\vec{A} + \nabla\cdot\left(\frac{1}{c}\partial_t \varphi + \nabla\cdot\vec{A}\right) = \frac{\vec{j}}{c}}$$

Then note that there is a constraint:

$$\partial_t \rho + \nabla\cdot\vec{j} = 0$$

So there are only three independent equations here, And we must specify a gauge condition in order to solve.

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① **Coulomb Gauge:**

$$\nabla \cdot \vec{A} = 0$$

Then

$$\boxed{\begin{aligned} -\nabla^2 \varphi &= \rho \\ -\Box \vec{A} &= \vec{j}/c + \frac{1}{c}\partial_t(-\nabla\varphi) \end{aligned}}$$

Often good for non-rel problems (Quasi-statics) And matter (ultra-relativistic plasma)

② **Covariant Gauge:**

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A} = 0$$

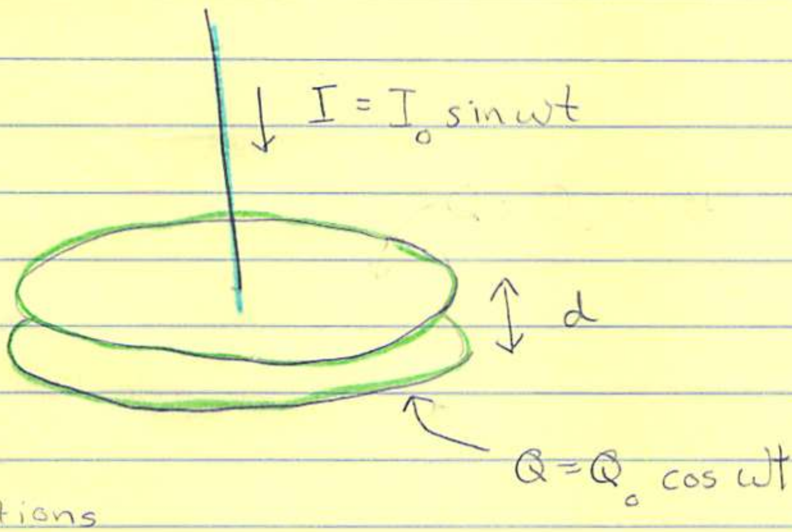
Then

$$\boxed{\begin{aligned} -\Box \varphi &= \rho \\ -\Box \vec{A} &= \vec{j}/c \end{aligned}}$$

Often a good choice for rel-problems with no preferred frame.

Capacitor pg. 1 (dimensional analysis)

An important example



Questions

1. What are the dimensionful parameters?

$$Q_0, \quad (d, z), \quad (\rho, R) \quad (\omega, c)$$

- What are the dimensionless parameters?

$$\frac{\omega R}{c} \ll 1 \quad \text{and} \quad \frac{d}{R}, \frac{z}{R} \ll 1 \quad \text{and} \quad \frac{\rho}{R}$$

So

small

$$E = \frac{Q}{R^2} f_E\left(\frac{\omega R}{c}, \frac{\rho}{R}\right) + \cancel{\frac{z}{R} g_E} + \cancel{\left(\frac{z}{R}\right)^2 h_E} \cdots$$

$$B = \frac{Q}{R^2} f_B\left(\frac{\omega R}{c}, \frac{\rho}{R}\right) + O(z/R)$$

Capacitor pg. 2 (dimensional analysis)

- So since $\omega R/c \ll 1$

$$E = \frac{Q}{R^2}\left[f_E^{(0)}\left(\frac{\rho}{R}\right) + \cancelto{0}{\left(\frac{\omega R}{c}\right) f_E^{(1)}}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)} + \cdots \right]$$

Simalrly

E is T-even
but ω is T-odd

$$B = \frac{Q}{R^2}\left[\cancelto{0}{f_B^{(0)}\left(\frac{\rho}{R}\right)} + \left(\frac{\omega R}{c}\right) f_B^{(1)} + \cancelto{0}{\left(\frac{\omega R}{c}\right)^2 f^{(2)}} + \cdots \right]$$

B is time reversal odd, but these are even

Summary of Dimensional Analysis

$$E = \frac{Q}{R^2}\left[f_E^{(0)}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)}\left(\frac{\rho}{R}\right) + \cdots \right]$$

$$B = \frac{Q}{R^2}\left[\left(\frac{\omega R}{c}\right) f_B^{(1)}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^3 f_B^{(3)} + \cdots \right]$$

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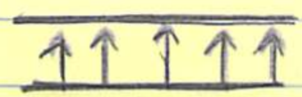
Ok How do we solve:

0th
$$\nabla \cdot E^{(0)} = 0$$
$$\nabla \times E^{(0)} = 0$$



$$E^{(0)} = \frac{Q_0}{\pi R^2} \cos \omega t \ \hat{z}$$

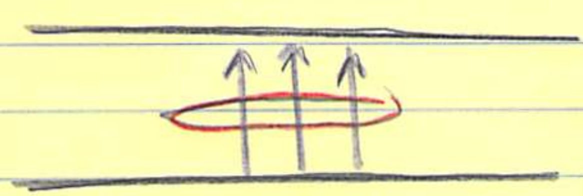
1st The $\nabla \times B^{(1)} = \frac{1}{c} \partial_t E^{(0)}$

2nd $- \nabla \times E^{(2)} = \frac{1}{c} \partial_t B^{(1)}$

These follow from
$$\nabla \times B = \frac{1}{c} \partial_t E$$
$$-\nabla \times E = \frac{1}{c} \partial_t B$$

1st Order

The displacement current $\equiv \partial_t E^{(0)}$ constrains $\vec{B}^{(1)}$:



$$\nabla \times B^{(1)} = \frac{1}{c} \partial_t E^{(0)}$$

$$\underbrace{\oint \vec{B} \cdot d\vec{l} = \frac{1}{c} \int \partial_t E^{(0)} \, 2\pi \rho \, d\rho}$$

or solve

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho B_\phi^{(1)} \right) = \frac{1}{c} \partial_t E_z^{(0)}$$

with:
$$E^{(0)} = \frac{Q_0}{\pi R^2} \cos \omega t$$

Solving this equation we find

$$B_\phi^{(1)} = -\frac{Q_0}{\pi R^2} \sin\omega t \left(\frac{\omega\rho}{2c}\right) + \cancelto{0}{\frac{C(t)}{\rho}}$$

In solving this equation we have discarded an irregular solution $\propto 1/\rho$. We see that $B_\phi^{(1)} \ll E_z^{(0)}$ since $\omega\rho/2c \ll 1$.

2nd Order

$$-\nabla \times E^{(2)} = \frac{1}{c}\, \partial_t B_\phi^{(1)}\, \hat{\phi}$$

Using the expression, $\nabla \times E = -\partial E^z/\partial\rho\, \hat{\phi}$, assuming that only E^z is non-zero, we have

$$+\frac{\partial E_z^{(2)}}{\partial\rho} = \frac{1}{c}\, \partial_t B_\phi^{(1)}$$

Integrating this expression we have

$$E_z^{(2)} = -\frac{Q_0 \cos\omega t}{\pi R^2} \frac{\omega^2\rho^2}{4c^2} + \text{Const}(t)$$

The constant is fixed by the fact that the total charge on the plate is $Q_0 \cos\omega t$.

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Integrating

$$Q(t) = \int_0^R 2\pi\rho \, d\rho \; \sigma(\rho, t)$$

$$Q_0 \cos\omega t = \int_0^R 2\pi\rho \, d\rho \left[E_z^{(0)} + E_z^{(2)} + \dots \right]$$

We find that Const (t) is $\frac{Q_0 \cos\omega t}{\pi R^2} \frac{\omega^2 R^2}{8c^2}$

Thus

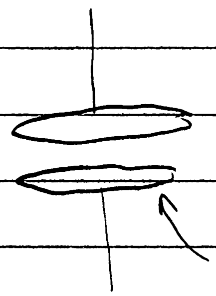
$$E^z(t,\rho) = \frac{Q_0 \cos\omega t}{\pi R^2} \left[1 + \frac{\omega^2 R^2}{8c^2} \left(1 - \frac{2\rho^2}{R^2}\right) + \dots \right]$$

$$B_\phi(t,\rho) = -\frac{Q_0 \sin\omega t}{\pi R^2} \left[\frac{\omega\rho}{2c} \right]$$

Notes:

- The second correction to E is of order $\omega^2 R^2 / c^2$ relative to $Q_0 / \pi R^2$
- The next correction to $\vec{B}$ is $\sim \left(\frac{\omega R}{c}\right)^3 \frac{Q_0}{\pi R^2}$, i.e it is $\sim \left(\frac{\omega R}{c}\right)^3$ smaller than $Q_0 / \pi R^2$

An important example (2nd Time) in Coulomb Gauge



$Q(t) = Q_0 \cos\omega t$

The equations of motion in the coulomb gauge are

$$-\nabla^2 \varphi = \cancelto{0}{\rho} \qquad\qquad \nabla\cdot A = 0$$

$$-\Box \vec{A} = \cancelto{0}{\vec{j}/c} + 1/c\, \partial_t(-\nabla\varphi)$$

where we have set $\rho = \vec{j}/c = 0$ since we are solving for the fields in between the plates. The fields are

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\partial_t \vec{A}$$

$$\vec{B} = \nabla \times \vec{A}$$

The boundary conditions are that $\vec{E}$ should be perpendicular to the plates, while $\vec{B}$ should be parallel to the plates

Solving the Laplace Equation for φ at zeroth order

0th :

$$\varphi = C_0(t) + C_1(t)\, z \qquad \vec{A} = 0$$

The coefficient $C_0(t)$ can be taken to be zero, and $C_1(t)$ must be adjusted to so that the charge on the plate must be $Q(t) = Q_0 \cos\omega t$ this fixes

$$\varphi = -\frac{Q_0 \cos\omega t}{\pi R^2}\, z$$

Actually this is the solution for φ to all orders. We will now set up an approximation scheme for $\vec{A}(t,\rho)$

Noting that the electric field must remain $\perp$ to the plate we must take $\vec{A}$ in the z-direction. Thus we try

$$\vec{A}(t,\rho) = A_z(t,\rho)\, \hat{z}$$

And note that the gauge condition is satisfied

$$\nabla \cdot A = 0$$

Then we approximate

$$\vec{A} \doteq \vec{A}^{(1)} + \vec{A}^{(2)} + \vec{A}^{(3)} + \dots$$

So we find from

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{1}{c}\partial_t(-\nabla\varphi)$$

The systems

1st

$$-\nabla^2\vec{A}^{(1)} = \frac{1}{c}\partial_t(-\nabla\varphi)$$

2nd

$$-\nabla^2 A^{(2)} = 0 \qquad \vec{A}^{(2)} = 0$$

3rd

$$-\nabla^2 A^{(3)} = -\frac{1}{c^2}\partial_t^2 A^{(1)}$$

⋮

So

So the first order equation reads

1st

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A^z}{\partial\rho} = -Q_0\frac{\sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)$$

Then solving this equation we find

$$A^z = -Q_0\frac{\sin\omega t}{\pi R^2}\left(\frac{-\omega\rho^2}{4c}\right) + A^z_{homo}$$

A^z_{homo} is a solution to the homogeneous equation

$$\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A^z_{homo}}{\partial\rho} = 0$$

Then the general solution to this equation is:

irregular at $\rho = 0$

$$A^z_{homo} = C_0(t) + C_1(t)\ln\rho$$

The $\ln\rho$ term can be discarded. The residual constant $C_0(t)$ is adjusted / interpreted with the charge on the plate per time

$$(\star)\quad Q(t) = \int_0^R 2\pi\rho\, d\rho\; E^z(\rho,t) = Q_0\cos\omega t$$

This yields

$$E^z(t,\rho) = -\nabla\varphi - \frac{1}{c}\partial_t \vec{A}$$

$$= \frac{Q_o \cos\omega t}{\pi R^2} - \frac{Q_o \cos\omega t}{\pi R^2}\left(\frac{\omega^2\rho^2}{4c^2}\right) + \frac{1}{c}\dot{C}_o(t)$$

So we find by demanding Eq ★ is satified

$$\boxed{A^z = \frac{Q_o \sin\omega t}{\pi R^2}\left(\frac{\omega\rho^2}{4c} - \frac{\omega R^2}{8c}\right)}$$

And thus we can compute E

$$B^{(1)} = \nabla\times A^{(1)} \Rightarrow B_\phi = -\frac{\partial A^z}{\partial\rho}$$

$$E = -\nabla\varphi - \frac{1}{c}\partial_t \vec{A}$$

$$\simeq E^{(0)} - \underbrace{\frac{1}{c}\partial_t \vec{A}^{(1)}}_{\text{second order}}$$

So as before

$$B^{(1)} = \frac{Q_o \sin\omega t}{\pi R^2}\left(\frac{\omega\rho}{2c}\right)$$

$$E = \frac{Q_o \cos\omega t}{\pi R^2}\left[1 + \frac{\omega^2 R^2}{8c^2}\left(1 - 2\rho^2/R^2\right) + \dots\right]$$