Magneto Statics

$$\left.\begin{aligned} \nabla \cdot E &= \rho \\ \nabla \times B &= j/c + \frac{1}{c}\partial_t E \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\frac{1}{c}\partial_t \vec{B} \end{aligned}\right\} \text{Full Maxwell eqs}$$

After the 1/c expansion

$$\left.\begin{aligned} \nabla \cdot E^{(0)} &= \rho \\ \nabla \times B^{(0)} &= 0 \\ \nabla \cdot B^{(0)} &= 0 \\ \nabla \times E^{(0)} &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} &\text{solve e-statics for } E^{(0)} \\ &B^{(0)} = 0 \end{aligned}$$

Now at next order

$$\left.\begin{aligned} \nabla \cdot E^{(1)} &= 0 \\ \nabla \times B^{(1)} &= j/c + \frac{1}{c}\partial_t E^{(0)} \\ \nabla \cdot B^{(1)} &= 0 \\ \nabla \times E^{(1)} &= 0 \end{aligned}\right\} \quad E^{(1)} \text{ is zero}$$

So find

calculated ~~with electro statics~~

$$\boxed{\begin{aligned} \nabla \times B^{(1)} &= \frac{j}{c} + \frac{1}{c}\frac{\partial \vec{E}^{(0)}}{\partial t} \\ \nabla \cdot B^{(1)} &= 0 \end{aligned}}$$

~~Maxwell displacement~~ current

The RHS consists of two known fens

$$j_{TOT} = \underset{\substack{\uparrow \\ \text{Current}}}{j} + \underset{\substack{\uparrow \\ \text{displacement} \\ \text{Current}}}{\frac{\partial E^{(0)}}{\partial t}}$$

Very often $\partial_t \vec{E}^{(0)}$ will be zero if there are only currents and no accumulation of charge. Then:

$$\vec{j}_{TOT} = \vec{j}$$

I will continue to write $\vec{j}$ ^ignoring $\partial_t \vec{E}^{(0)}$. But, the dérivations would go through, if I used $\vec{j}_{TOT}$ instead of $\vec{j}$.

Then magnetostatics is:

This is equivalent to Ampères Law and Biot-Savat Law $\Longrightarrow$

$$\begin{cases} \nabla \times B = j/c \\ \nabla \cdot B = 0 \\ F = q\frac{\vec{v}}{c} \times \vec{B} \quad \text{and} \quad d\vec{F} = \frac{I}{c}\, d\vec{l} \times \vec{B} \end{cases}$$

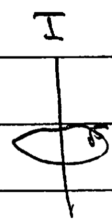
Ampères Law

$$\oint B \cdot dl = \int \vec{\nabla} \times \vec{B} \cdot d\vec{a}$$

$$= \int d\vec{a} \cdot \left(\frac{\vec{j}}{c}\right)$$

$$\boxed{\oint B \cdot dl = \frac{I}{c}}$$

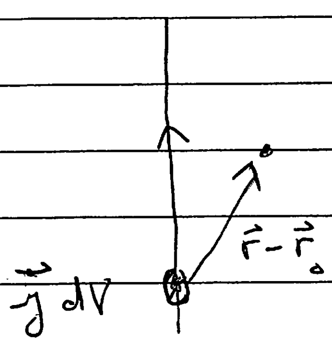
Ex



$$2\pi\rho B = I/c \longrightarrow B = \frac{I/c}{2\pi\rho}$$

Biot Savat

$$\vec{B} = \int d^3\vec{r}\ \vec{j} \times \left[\frac{(\vec{r}-\vec{r}_0)}{4\pi |\vec{r}-r_0|^3}\right]$$

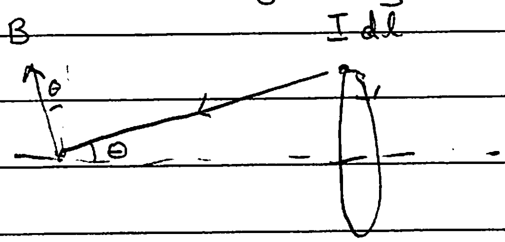


For a wire

$$\vec{B} = \int \frac{I}{c} d\vec{l} \times \frac{(\vec{r}-\vec{r}_0)}{4\pi |\vec{r}-\vec{r}_0|^2}$$

- We will derive this in the next section

Ex: A ring of current. Calculate $\vec{B}$ along x-axis



$$dB_x = dB\sin\theta = \frac{I/c\, dl \sin\theta}{4\pi\,(x^2+a^2)} = \frac{I/c\, dl\, a}{4\pi\,(x^2+a^2)^{3/2}}$$

$$B_x = \frac{(I/c\, \pi a^2)}{4\pi\,(x^2+a^2)^{3/2}} \cdot 2$$

$$B_x = \frac{m}{4\pi\,(x^2+a^2)^{3/2}} \cdot 2 \qquad \underbrace{m = (I\,\pi a^2)/c}_{\text{magnetic moment}}$$

$$\boxed{B_z \xrightarrow[x\to\infty]{} \frac{m}{4\pi x^3} \cdot 2}$$

we will describe the two later when we describe the multipole expansion

Vector Potential and a proof of Biot-Savat

$$\nabla \times B = \vec{j}/c$$

$$\nabla \cdot B = 0$$

- Helmholtz theorem: whenever $\nabla \cdot \vec{B} = 0$ $\vec{B}$ can be written as a curl of a vector field

$$\boxed{B = \nabla \times \vec{A}}$$

However, the choice is not unique:

$$\boxed{A \rightarrow \vec{A} + \nabla \Lambda} \leftarrow \text{Gauge Transform}$$

leads to the same B-field since $\vec{\nabla} \times (\vec{\nabla}\Lambda) = 0$, $\varepsilon^{ijk} \partial_j \partial_k \Lambda = 0$.

- Plugging $\vec{B} = \nabla \times \vec{A}$ into $\nabla \times B = j/c$

$$\nabla \times (\nabla \times A) = j/c$$

or using vector identities:

$$-\nabla^2 \vec{A} + \vec{\nabla}(\nabla \cdot \vec{A}) = \vec{j}/c$$

Aside

$$(\nabla \times (\nabla \times A))^i = \varepsilon^{ijk} \partial_j (\varepsilon_{klm} \partial^l A^m) = \varepsilon^{ijk} \varepsilon_{klm} \partial_j \partial^l A^m$$

$$= (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) \partial_j \partial^l A^m$$

$$= \partial^i (\partial_j A^j) - \partial_l \partial^l A^i = \left[\vec{\nabla}(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right]^i$$

Vector Potential & Bio-Savat

So if we adopt the Coulomb Gauge

$$\vec{\nabla}\cdot\vec{A} = 0$$

Which fixes the choice of Λ, then

$$-\nabla^2\vec{A} = \vec{j}/c$$

Thus the cartesian components of $\vec{A} = (A^x, A^y, A^z)$ obey the Laplace eqn; e.g.

$$-\nabla^2 A^x = j^x/c$$

So we write down the solution straight away

$$\boxed{\vec{A}(\vec{r}) = \int d^3x \; \frac{\vec{j}/c\,(\vec{x})}{4\pi\,|\vec{r}-\vec{x}|}}$$

kind of coulomb law for magnetic fields

To compute $\vec{B}$ we take the curl:

$$\vec{B} = \nabla_r \times A(\vec{r}) = \int d^3\vec{x} \; \vec{\nabla}_r \times \left(\frac{\vec{j}(x)/c}{4\pi\,|\vec{r}-\vec{x}|} \right)$$

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For a constant vector $\vec{j}(x)$ (indep of $\vec{r}$)

$$\vec{\nabla} \times \left(\vec{j}\, \phi(r) \right) = \vec{j} \times \left(-\vec{\nabla}_r \phi \right)$$

Yield

gradient of $\frac{1}{r}$ potential $= \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$

$$\vec{B} = \int d^3\vec{x}\ \vec{j}(x) \times \left(-\nabla \frac{1}{4\pi |\vec{r} - \vec{x}|} \right)$$

$$\boxed{\vec{B} = \int_V d^3 x\ \vec{j}(x) \times \frac{(\vec{r} - \vec{x})}{4\pi |r - x|^3}}$$

Biot-Savat Law