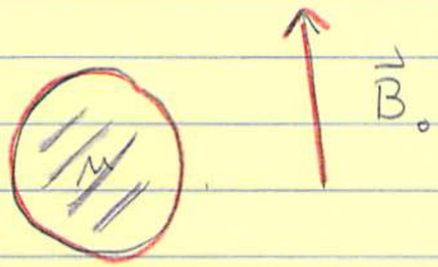
Magnetic Scalar Pot pg. 1

Example:



Now we will solve again for the a magnetized sphere placed in an external magnetic field of magnitude B_0. But we will use the *magnetic scalar potential* to solve the problem

- The magnetic field induces a magnetic moment inside the material

$$\nabla \times H = \vec{J}_{ext}$$

$$B = \mu H$$

$$\nabla \cdot B = 0$$

- In any current free region, $\nabla \times H = 0$
Then we can introduce a magnetic scalar potential, Ψ

$$\boxed{\vec{H} = -\vec{\nabla}\Psi}$$

- Then inside the sphere:

Relating Ψ_{in} and Ψ_{out} is different from e-statics
see below

$$\vec{\nabla} \cdot \left(\frac{-\nabla \Psi_{in}}{\mu} \right) = 0$$

assume $\mu = \text{const}$
(does not apply accross jump)

$$-\nabla^2 \Psi_{in} = 0$$

- Similarly outside the sphere, $-\nabla^2 \Psi_{out} = 0$

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- Then inside the sphere

$$\Psi^{in} = \sum_\ell \left(\tilde{A}_\ell r^\ell + \cancel{\frac{\tilde{B}_\ell}{r^{\ell+1}}} \right) P_\ell(\cos\theta)$$

$$\Psi^{out} = \sum_\ell \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

- Inside, regularity forces $\tilde{B}_\ell = 0$
- Outside, the fact that $B^{out} \xrightarrow[r\to\infty]{} B_0$, leads to

$$\Psi_m \xrightarrow[r\to\infty]{} -B_0 \, r\cos\theta \qquad C_1 = -B_0 \quad \text{all other } 0$$

- Now then we need boundary conditions

$$\vec{n} \times (\vec{H}_{out} - \vec{H}_{in}) = \cancel{K_{free}/c}$$

$$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0 \implies \vec{n} \cdot \left(\vec{H}_{out} - \mu \vec{H}_{in} \right)$$

- So

$$\vec{H} = -\nabla \Psi_m = -\frac{\partial \Psi}{\partial r} \hat{r} + -\frac{1}{r} \frac{\partial \Psi}{\partial \theta} \hat{\theta}$$

Then

$$\Psi_{out} = -B_0 r\cos\theta + \sum_\ell \frac{D_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

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- So

$$H^r_{out} = +B_0 \cos\theta + \sum_\ell +(\ell+1)\frac{D_\ell}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$H^\theta_{out} = -B_0 \sin\theta + \sum_\ell -\frac{D_\ell}{r^{\ell+2}} \frac{dP_\ell}{d\theta}(\cos\theta)$$

While

$$H^r_{in} = -\sum_\ell \ell A_\ell r^{\ell-1} P_\ell(\cos\theta)$$

$$H^\theta_{in} = -\sum_\ell A_\ell r^{\ell-1} \frac{dP_\ell}{d\theta}(\cos\theta)$$

- Limitting ourselvele to $\ell=1$:

$$H^r_{out} = +B_0 \cos\theta + 2\frac{D}{r^3}\cos\theta$$

$$H^\theta_{out} = -B_0 \sin\theta + \frac{D}{r^3}\sin\theta$$

In

$$H^r_{in} = -A\cos\theta$$

$$H^\theta_{in} = +A\sin\theta$$

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Then from: $\swarrow$ $B^{\perp}$ is continuous

$$\left. H^r_{out} - \mu H^r_{in} = 0 \right|_{r=a} \implies B_0 + \frac{2D}{a^3} + \mu A = 0$$

and $\swarrow$ $H^{\parallel}$ is continuous, since $\vec{K}_{ext} = 0$

$$\left. H^\theta_{out} - H^\theta_{in} = 0 \right|_{r=a} \implies -B_0 + \frac{D}{a^3} - A = 0$$

- Solving gives:

$$D = B_0 a^3 \frac{(\mu-1)}{(\mu+2)} \qquad A = -\frac{3B_0}{2+\mu}$$

- Further one can check that for $l \neq 1$ the eqs are trivially satisfied by $A_l = D_l = 0$. Thus the full solution is

$$\psi^{in}_m = -\frac{3B_0}{2+\mu} r\cos\theta$$

$$\psi^{out}_m = -B_0 r\cos\theta + \overbrace{\frac{B_0 a^3}{r^2}\frac{(\mu-1)}{(\mu+2)}\cos\theta}^{\text{this takes the form of a dipole potential}}$$

or

$$\vec{H}^{in} = \frac{3B_0}{2+\mu}\hat{z}$$ $\nwarrow$ constant field

$\swarrow$ dipole field

$$\vec{H}^{out} = B_0\hat{z} + \left[3(\vec{n}\cdot\vec{m})\vec{n} - \vec{m}\right]/4\pi r^3$$

Mag Scalar 7 & B.C.

where,

$$\boxed{\vec{m} = 4\pi B_0 a^3 \left(\frac{\mu-1}{\mu+2}\right).}$$

Now lets check that Boundary Conditions Are Satisfied

- The surface current is

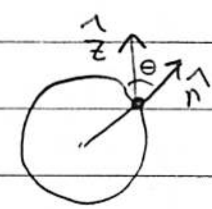
$$\vec{n} \times (\vec{M}_{out} - \vec{M}_{in}) = \vec{K}_{mat}/c$$

Now the magnetization outside the sphere is $= 0$ since we have no medium outside. Thus

$$-\vec{n} \times \vec{M}_{in} = \vec{K}_{mat}/c$$

- Then $\vec{M} = (\mu - 1)\vec{H}$, so

$$\frac{\vec{K}_{mat}}{c} = -\vec{n} \times \left[(\mu-1) \frac{3B_0}{2+\mu} \hat{z} \right]$$



$$\frac{\vec{K}_{mat}}{c} = (\mu-1)\frac{3B_0}{2+\mu} \underbrace{(-\vec{n} \times \hat{z})}_{\hat{\phi}}$$

see picture

$$\boxed{\frac{\vec{K}_{mat}}{c} = 3B_0 \left(\frac{\mu-1}{\mu+2}\right) \sin\theta \, \hat{\phi}}$$

Mag-Scalar B.C.

- Thus, we see that the current distribution on the surface of the sphere:

$$\vec{K} \propto \sin\theta \, \hat{\phi},$$

is the same as for the rotating charged sphere, and thus the the induced magnetic fields in this case are the same (up to constant) as for the rotating charged sphere.