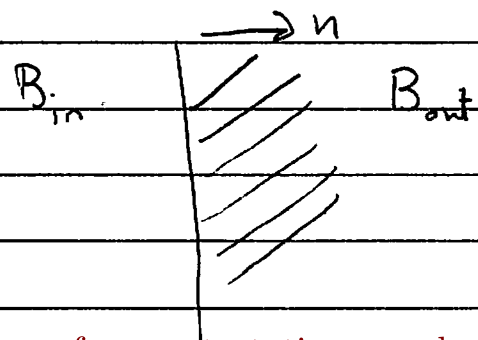
Last Times

$$\nabla \times B = j/c$$

$$\nabla \cdot B = 0$$

Wrote Down B.C

Boundary conditions are derived on the next page



$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \vec{K}/c$$

$$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$$

The equations of magnetostatics can also be written down for $\boldsymbol{A}$

This can also be written for $\vec{A}$

$$-\nabla^2 \vec{A} = \vec{j}/c \qquad \vec{B} = \nabla \times \vec{A}$$

Question:

- What is the vector potential corresponding to a const $\vec{B}$ field?

Ans:

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

Prf

$$(\nabla \times A)_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \frac{B^l r^m}{2}$$

$$= \varepsilon_{ijk} \varepsilon_{lmk} \delta^m_j \frac{B^l}{2} = \overbrace{\varepsilon_{ijk} \varepsilon_{ljk}}^{2\delta_{il}} \frac{B^l}{2} = B_i$$

Boundary Value Problems in Magneto Statics

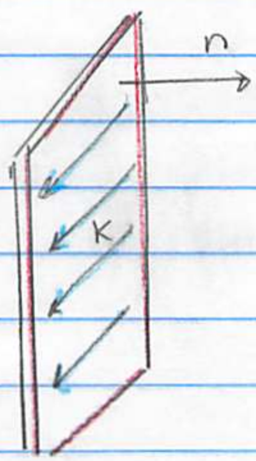
$$\nabla \times B = \vec{j}/c \qquad (1)$$

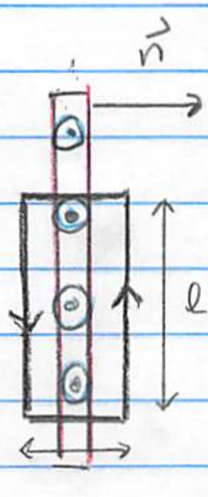
$$\nabla \cdot \vec{B} = 0 \qquad (2)$$

Need Boundary Conditions

Then to find the $\parallel$ boundary conditions we integrat (1)

$$\oint \vec{B} \cdot d\vec{l} = \int \vec{j} \cdot d\vec{a}$$





$$(B^{\parallel}_{out} - B^{\parallel}_{in})\, l = K\, l$$

$K \equiv$ Current per length $= j\, \Delta h$

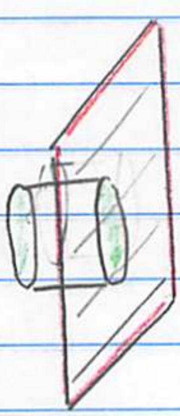
$$B^{\parallel}_{out} - B^{\parallel}_{in} = K$$

Or using a coordinate free notation:

$$\boxed{n \times (\vec{B}_{out} - \vec{B}_{in}) = \vec{K}}$$

Similarly,

$$\int_{\text{pill-box}} \nabla \cdot B = A\, \vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$$



$$\boxed{\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0}$$

or

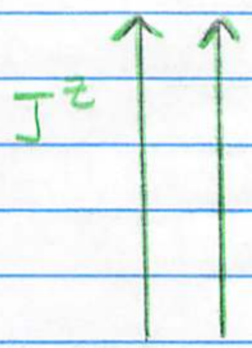
$$B_{\perp}^{out} - B_{\perp}^{in} = 0$$

Solving For the Magnetic Field

1. Direct use of $-\nabla^2 \vec{A} = \vec{J}$. In general very complicated except of problems with symmetry:

- First write down $-\nabla^2 \vec{A}$ in different coordinates (or look it up on Wikipedia!) It's a mess!

- If the currents have only $\hat{z}$ components Then $\vec{A}$ has only z components



$$\vec{A}(\rho,\phi) = A^z(\rho,\phi)\,\hat{z}$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{-1}{\rho^2}\frac{\partial}{\partial\phi^2}\right] A^z = J^z$$

homework

Now use separation of variables

- Similarly if $\vec{J}$ is azimuthally symmetric

$$\vec{J} = J_\phi(r,\theta)\,\hat{\phi} \quad \text{implies} \quad \vec{A} = A_\phi(r,\theta)\,\hat{\phi}$$

Then writing $-\nabla^2 \vec{A} = \vec{J}$ gives with $A_\phi(r,\theta)$

$$-\nabla^2 A_\phi(r,\theta) + \frac{A_\phi}{r^2 \sin^2\theta} = J_\phi(r,\theta)$$

Now use again separation in homework

(2) In the absence of boundaries directly integrate

$$\vec{B}(\vec{r}) = \int d^3\vec{x} \; \frac{\vec{J}(\vec{x})}{4\pi|\vec{r}-\vec{x}|}$$

We will do this in the next example.

(3) For some cases one can use a magnetic scalar potential Ψ_m. This is often technically the simplest.

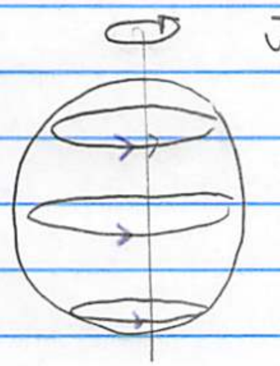
We will do this for an example problem later

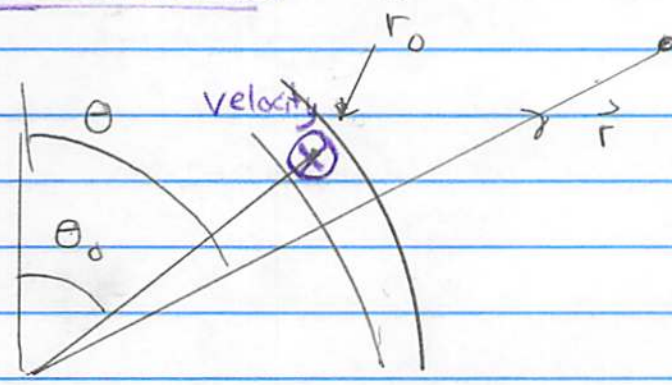
- All of these methods will use the b.c.

$$\boxed{\begin{aligned} \vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) &= \vec{K}/c \\ \vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) &= 0 \end{aligned}}$$

Rotating Charged Spherical Shell

$\vec{A}(\vec{r}) = ?$





- A charged shell of charge Q and radius a, rotates with angular velocity $\vec{\omega}$

$$\sigma = \frac{Q}{4\pi a^2} \qquad \vec{v} = \omega a \sin\theta_0 \, \hat{\phi}_0 = \vec{\omega} \times \vec{r}_0$$

- So the current density on an element of surface is

$$\vec{j} = \sigma \frac{\vec{v}}{c} \, \delta(r_0 - a)$$

$$\vec{j}(r_0, \theta_0, \phi_0) = \sigma \frac{\omega a}{c} \delta(r_0 - a) \sin\theta_0 \, \hat{\phi}_0$$

- The vector potential will follow the current and be azimuthally symmetric

- We will find $\vec{A}(\vec{r})$ outside the shell

Rotating Sphere 2

- Direct Integration of magnetic coulomb law determines the vector potential:

$$\vec{A}(\vec{r}) = \int d^3r_0 \, \frac{\vec{j}(\vec{r}_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

is a good choice here:

$$(\star) \quad A(\vec{r}) = \int r_0^2 dr_0 \, d\Omega_0 \, \frac{\sigma \, \delta(r_0 - a) \, \omega a \sin\theta_0 / c \;\hat{\phi}_0}{4\pi |\vec{r} - \vec{r}_0|}$$

- Now look outside sphere $r > r_0$

$$\star\star \quad \frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{lm} \frac{1}{(2l+1)} \frac{r_0^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta_0, \phi_0)$$

where $\vec{r}_0 = (r_0 \sin\theta_0 \cos\phi_0, r_0 \sin\theta_0 \sin\phi, \cos\phi_0)$. Then

- Look at

$$\sin\theta_0 \hat{\phi}_0 = + \underbrace{\sin\theta_0 \cos\phi_0}_{\propto Y_{11} \text{ and } Y_{1-1}} \hat{y} - \underbrace{\sin\theta_0 \sin\phi_0}_{\propto Y_{11} \text{ and } Y_{1,-1}} \hat{x}$$

- Substituting Eq. $(\star\star)$ into $\star$ we find integrals like:

$$\int d\Omega_0 \, Y^*_{lm}(\theta_0, \phi_0) \, (\sin\theta_0 \cos\phi_0) = 0 \quad \text{unless} \quad l = 1$$

Thus only $l = 1$ survives integrations $\int d\Omega_0$.

• Now recognize that for any function $f(\theta_0, \phi_0)$ lying in the span of Y_{1m} (an $l=1$ function), we have

$$\sum_m \int d\Omega_0 \, Y_{1m}(\theta, \phi) \, Y^*_{1m}(\theta_0, \phi_0) \, f(\theta_0 \phi_0) = f(\theta, \phi)$$

And thus

$$\sum_m \int d\Omega_0 \, Y_{1m}(\theta, \phi) Y^*_{1m}(\theta_0, \phi_0) \, (\sin\theta_0 \, \hat{\phi}_0) = \sin\theta \, \hat{\phi}$$

• Yielding from ☆ and ☆☆

$$\vec{A}(r) = \frac{\sigma}{c} \frac{w a^4}{3 r^2} \sin\theta . \hat{\phi}$$

we used $2l+1 = 3$, and $\int r_0^3 \, \delta(r_0 - a) = a^3$

Now $\omega \sin\theta \, \hat{\phi} = \vec{\omega} \times \hat{r}$ by geometry, so with $\sigma = \frac{Q}{4\pi a^2}$ we find

$$\vec{A} = \frac{\overbrace{(Qa^2/3c \, \vec{\omega})}^{\equiv \vec{m}} \times \hat{r}}{4\pi r^2} \qquad \vec{m} \equiv Qa^2/3c \, \vec{\omega}$$

So outside the sphere we have:

$$\vec{A} = \frac{\vec{m} \times \vec{r}}{4\pi r^3}$$ which is a simple dipole

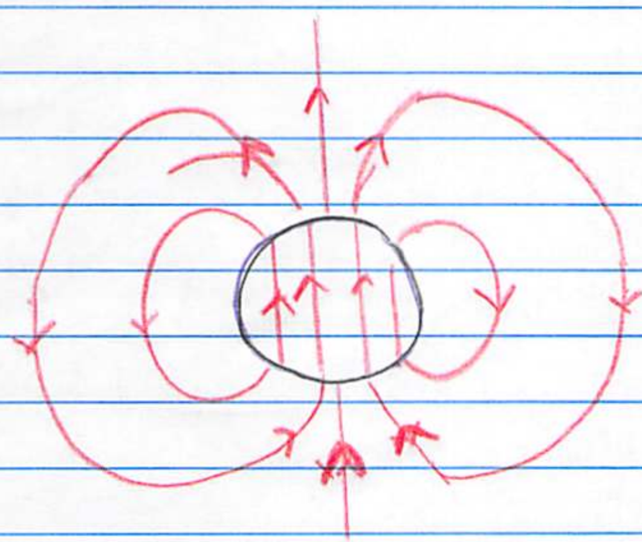
$$\vec{B} = \frac{3\vec{n}(\vec{n} \cdot \vec{m}) - \vec{m}}{4\pi r^3}$$

- An entirely similar calculation (which interchanges $r_>$ and $r_<$) determines the vector potential inside the sphere

$$\vec{A} = \frac{\vec{m} \times \vec{r}}{4\pi a^3}$$

this is the vector potential of a constant field. From the start of lecture $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$ and so

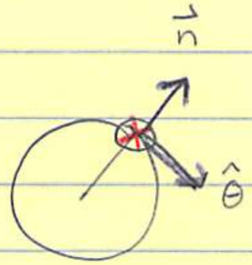
$$\vec{B} = \frac{2}{3} \frac{\vec{m}}{(4\pi/_3 \, a^3)}$$



Rotating Sphere - Check BC

- We will now check that the B.C. are satisfied.

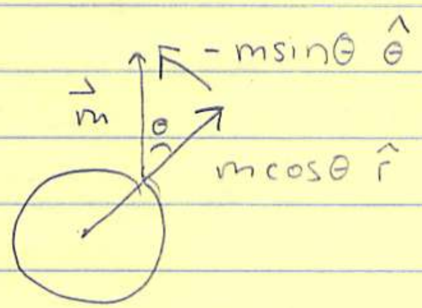
$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \frac{\vec{K}}{c}$$



$$(B_{out})_\theta - (B_{in})_\theta = \frac{K_\phi}{c}$$

into page

- Using:



$$\vec{m} = m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta}$$

$$B^\theta_{out} = \frac{1}{4\pi a^3}(+m\sin\theta)$$

this comes from $-\vec{m}$ of $\frac{3(n\cdot\vec{m})\vec{n} - \vec{m}}{4\pi r^3}$

$$B^\Theta_{in} = \frac{2(-m\sin\theta)}{4\pi a^3}$$

This is from $\vec{m}$ in $\vec{B}_{in} = 2\vec{m}/4\pi r^3$

- So

$$B^\theta_{out} - B^\theta_{in} = \frac{3m\sin\theta}{4\pi a^3}$$

$$m = \frac{Qa^2\omega}{3c} \qquad \sigma = \frac{Q}{4\pi a^2}$$

$$= \sigma\omega a\sin\theta$$

surface current ✓