B Scalars, Vectors, Tensors

(a) We will use the Einstein summation convention

$$\boldsymbol{V} = V^1\boldsymbol{e}_1 + V^2\boldsymbol{e}_2 + V^3\boldsymbol{e}_3 = V^i\boldsymbol{e}_i \tag{B.1}$$

Here repeated indices are implicitly summed from $i = 1\ldots 3$, where $1,2,3 = x,y,z$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors in the x, y, z directions.

(b) Under a rotation of coordinates the coordinates change in the following way

$$x^i = R^i_{\ j} x^j \,. \tag{B.2}$$

where R we think of as a rotation matrix, where i labels the rows of R and j labels the columns of R.

(c) Scalars, vectors and tensors are defined by how there components transform

$$S \to \underline{S} = S \,, \tag{B.3}$$
$$V^i \to \underline{V}^i = R^i_{\ j} V^j \,, \tag{B.4}$$
$$T^{ij} \to \underline{T}^{ij} = R^i_{\ \ell} R^j_{\ m} T^{\ell m} \,. \tag{B.5}$$

We think of upper indices (contravariant indices) as row labels, and lower indices (covariant indicies) as column labels. Thus V^i is thought of as column vector

$$V^i \leftrightarrow \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{B.6}$$

labelled by V^1, V^2, V^3 – the first row entry, the second row entry, the third row entry. Contravariant means "opposite to coordinate vectors" $\boldsymbol{e}_i$ (see next item)

(d) Under a rotation of coordinates the basis vectors also transform with

$$\underline{\boldsymbol{e}}_i \to \underline{\boldsymbol{e}}_i (R^{-1})^i_{\ j} \tag{B.7}$$

This transformation rule is how the lower (or covariant) vectors transform. The covariant components of a vector V_i transform as

$$(\underline{V}_1 \underline{V}_2 \underline{V}_3) = (V_1 V_2 V_3)\Big(R^{-1}\Big) \,. \tag{B.8}$$

covariant means "the same as coordinate vectors", *i.e.* with R^{-1} but as a row.

(e) Since $R^{-1} = R^T$ there is no need to distinguish covariant and contravariant indices for rotations. This is not the case for more general groups.

(f) With this notation the vectors and tensors (which are physical objects)

$$\underline{\boldsymbol{V}} = \underline{V}^i \underline{\boldsymbol{e}}_i \quad = \quad V^i \boldsymbol{e}_i = \boldsymbol{V} \tag{B.9}$$
$$\underline{\boldsymbol{T}} = \underline{T}^{ij} \underline{\boldsymbol{e}}_i \underline{\boldsymbol{e}}_j \quad = \quad T^{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{T} \tag{B.10}$$

are invariant under rotations, but the components and basis vectors change.

(g) Vector and tensor components can be raised and lowered with δ^{ij} which forms the identity matrix,

$$\delta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.11}$$

i.e.

$$V^i = \delta^{ij} V_j \tag{B.12}$$

We note various trivia

$$\delta^i{}_i = 3 \qquad \delta_{ij}\delta^{ij} = 3 \quad \delta_{ij}\delta^{jk} = \delta^k_i \tag{B.13}$$

(h) The epsilon tensor ϵ^{ijk} is

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} \pm 1 & \text{for } i,j,k \text{ an even/odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases} \tag{B.14}$$

For example, $\epsilon^{123} = \epsilon^{312} = \epsilon^{231} = 1 = \epsilon_{123} = 1$ while $\epsilon^{213} = -\epsilon^{123} = -1$.

i) The epsilon tensor is useful for simplifying cross products

$$(\boldsymbol{a} \times \boldsymbol{b})^i = \epsilon^{ijk} a_j b_k \tag{B.15}$$

ii) A useful identity is

$$\epsilon^{ijk}\epsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl} \tag{B.16}$$

which can be used to deduce the "b(ac) - (ab)c" rule for cross products

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{B.17}$$

iii) The "b(ac) - (ab)c" rule arises a lot in this course and is essential to deriving the wave equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \tag{B.18}$$

and to identifying the transverse pieces of a vector. For instance the component of a vector $\boldsymbol{v}$, transverse to a unit vector $\boldsymbol{n}$, is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{v}) = \boldsymbol{v}_T = -(\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} + \boldsymbol{v} \tag{B.19}$$

(i) Derivatives work the same way. $\partial_i \equiv \frac{\partial}{\partial x^i}$. With this notation we have

$$\nabla \cdot \boldsymbol{E} = \partial_i E^i \tag{B.20}$$

$$(\nabla \times \boldsymbol{E})^i = \epsilon^{ijk} \partial_j E_k \tag{B.21}$$

$$(\nabla \phi)_i = \partial_i \phi \tag{B.22}$$

$$(\nabla^2 \phi) = \partial_i \partial^i \phi \tag{B.23}$$

(B.24)

and expressions like

$$\partial_i x^j = \delta^j_i \qquad \partial_i x^i = d = 3 \tag{B.25}$$

(j) A general second rank tensor T^{ij} is decomposed into its irreducible components as

$$T^{ij} = \mathring{T}^{ij}_S + \epsilon^{ijk} V_k + \tfrac{1}{3} T^\ell{}_\ell \delta^{ij} \tag{B.26}$$

where $\mathring{T}^{ij}_S = \frac{1}{2}(T^{ij} + T^{ji} - \frac{2}{3} T^\ell{}_\ell \delta^{ij})$ is a *symmetric-traceless* component of T^{ij} and V_k is a vector associated with the antisymmetric part of T^{ij}, $V_k = \frac{1}{2}\epsilon_{k\ell m} T^{\ell m}$.

(k) We will discussed how to reduce a tensor integral into a set of scalar integrals later in this course, e.g.

$$\int d^3\boldsymbol{x}\; x^i x^j x^\ell x^m\, f(x) = \left[\frac{4\pi}{15} \int_0^\infty dx\, x^6 f(x) \right] \left(\delta^{ij}\delta^{\ell m} + \delta^{i\ell}\delta^{jm} + \delta^{im}\delta^{j\ell} \right) \tag{B.27}$$

Here $x = |\boldsymbol{x}|$ denotes the norm of the vector $\boldsymbol{x}$. Thus, $f(x)$ denotes a function of the radius, $f(\sqrt{x_1^2 + x_2^2 + x_3^2})$.