Vectors and Tensors

- We will use a new notation for vectors

$$\vec{V} = V^1 \vec{e}_1 + V^2 \vec{e}_2 + V^3 \vec{e}_3 = \sum_{i=1}^{3} V^i \vec{e}_i$$

where $(V^1, V^2, V^3) \equiv (V^x, V^y, V^z)$ and $(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (\hat{\imath}, \hat{\jmath}, \hat{k})$ are the x, y, z components and corresponding unit vectors

Then we use a summation convention, where repeated indices are summed over

$$\vec{V} = V^i \vec{e}_i \qquad (\text{same as } \vec{V} = \sum_i V^i \vec{e}_i$$

- Vectors are physical objects:

If the coordinates are rotated $\vec{V}$ remains unchanged. But, the components V^i are changed. The figure shows how V^y is changed and how the basis vectors are also changed

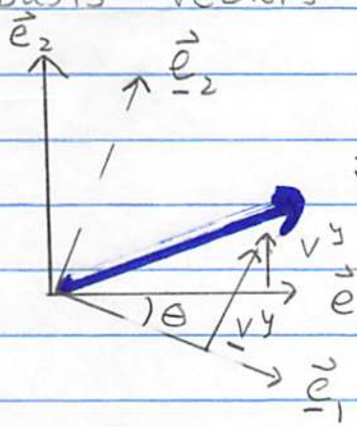
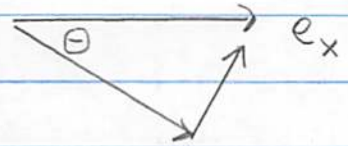
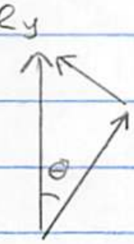


Figure 1.

From the picture:





$$e_x = \underline{e}_x \cos\theta + \sin\theta \, \underline{e}_y$$

See pictures

$$e_y = \underline{e}_y \cos\theta - \sin\theta \, \underline{e}_x$$

Thus since the vector is unchanged:

$$\vec{V} = V^x \vec{e}_x + V^y \vec{e}_y = \underline{V}^x \vec{\underline{e}}_x + \underline{V}^y \vec{\underline{e}}_y \quad , \text{ or}$$

$$\left.\begin{aligned} \vec{V} = {} & (V^x \cos\theta - \sin\theta \, V^y)\, \underline{e}_x \\ & + (V^x \sin\theta + V^y \cos\theta)\, \underline{e}_y \end{aligned}\right\} \text{ i.e. } \quad \begin{aligned} \underline{V}^x &= V^x \cos\theta - V^y \sin\theta \\ \underline{V}^y &= V^x \sin\theta + V^y \cos\theta \end{aligned}$$

In general:

$$\boxed{\underline{V}^i = r^i{}_j \, V^j} \qquad \text{rotation rule}$$

Where $r^i{}_j$ are the elements of a rotation matrix,

$$(\mathcal{R})^i{}_j = r^i{}_j$$

elements of $\mathcal{R}$

matrix $\mathcal{R}$

- Think of v^i as column vector $\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = \begin{pmatrix} v^i \\ \cdot \\ \cdot \end{pmatrix}$

Then the rotated vector is

$$\begin{pmatrix} \underline{v}^i \\ \\ \\ \end{pmatrix} = \begin{pmatrix} r^1{}_1 & r^1{}_2 & r^1{}_3 \\ r^2{}_1 & r^2{}_2 & r^2{}_3 \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} v^j \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

For the rotation shown on fig 1. we have

$$(\mathcal{R})^i{}_j = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ \hline & & 1 \end{array}\right)$$

- <u>Notation</u>:

$\mathcal{R}$ the matrix

$$(\mathcal{R})_{ij} = (\mathcal{R})^i{}_j = (\mathcal{R})_i{}^j = (\mathcal{R})^i{}_j = r^i{}_j$$

The placement of indices around $\mathcal{R}$ is irrelevant. the first index is the row, the second index is the column. We make a distinction between the matrix $\mathcal{R}$ (which has inverse, transpose, etc) and the entries $r^i{}_j$ which are just numbers

In terms of matrices

$$\underline{v} = \mathcal{R}\, v$$

column vector

rotation matrix

column vector

- $\mathcal{R}$ is an orthogonal matrix $\mathcal{R}^T = \mathcal{R}^{-1}$.

We note that the norm of the vector is invariant under rotation

$$\|\vec{v}\| = v^T v = \underline{v}^T \underline{v} = \begin{pmatrix} v^x & v^x \end{pmatrix} \begin{pmatrix} v^x \\ v^x \end{pmatrix}$$

$$= (v^x)^2 + (v^y)^2$$

Thus since $\underline{v} = \mathcal{R} v$ and $\underline{v}^T = v^T \mathcal{R}^T$ we have

$$\underline{v}^T \underline{v} = v^T \underbrace{\mathcal{R}^T \mathcal{R}}_{\mathbb{1} \text{ identity}} v = v^T v$$

- We now discuss that lower indices transform under the rotation according to the inverse representation and as a row (i.e according to the "inverse-transpose" representation)

For example the basis vectors $\vec{e}_i$ transform as

$$\underline{\vec{e}}_i = \vec{e}_i \, (\mathcal{R}^{-1})^i{}_j$$

Think of basis vectors, and more generally lower indices (also known as covariant indices), as a row, e.g.

$$(\underline{\vec{e}}_1 \;\; \underline{\vec{e}}_2 \;\; \underline{\vec{e}}_3) = (\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3) \begin{pmatrix} \\ \mathcal{R}^{-1} \\ \\ \end{pmatrix}$$

In this way, $\vec{V}$ is unchanged under rotation, e.g.

$$\underline{\vec{V}} = \underline{\vec{e}}_i \, \underline{V}^i$$

$$= (e_i \ldots) \underbrace{\begin{pmatrix} \mathcal{R}^{-1} \\ \\ \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \\ \end{pmatrix}}_{\mathbb{1} \;\leftarrow\; \text{identity matrix}} \begin{pmatrix} V^i \\ \vdots \end{pmatrix}$$

$$= \vec{e}_i \, V^i = \vec{V}$$

We used

$$(\mathcal{R}^{-1})^i{}_j \, (\mathcal{R})^j{}_k = \delta^i{}_k \equiv \left\{ \begin{array}{l} \text{Identity} \\ \text{matrix elements} \end{array} \right. \begin{array}{l} = 1 \text{ if } i = k \\ \;\; 0 \text{ otherwise} \end{array}$$

Dual Basis

Covariant versus Contravariant Basis Vectors:

- For every basis of covariant (lower) basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ look for a set of contra-variant (upper) basis vector $(\vec{e}^1, \vec{e}^2, \vec{e}^3)$ which satify

$$\star \qquad \vec{e}^i \cdot \vec{e}_j = \delta^i{}_j \qquad (\text{Eq} \star)$$

Clearly for a cartesian orthonormal coordinate system where $(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (\hat{x}, \hat{y}, \hat{z})$ we must take $(\vec{e}^1, \vec{e}^2, \vec{e}^3) = (\hat{x}, \hat{y}, \hat{z})$ so upper and lower indexed basis vectors are the same in this simple case, but will not be for more general basis sets

- We may expand $\vec{V}$ in the upper indexed (contra-variant) basis set, or the lower one:

$$\vec{V} = V_i \vec{e}^i = V^i \vec{e}_i$$

In this case upper and lower are the same:

$$V_x = V^x \qquad \text{or} \qquad V_i = \delta_{ij} V^j \qquad V^i = \delta^{ij} V_j$$

So indices are raised and lowered with δ^{ij} + δ_{ij}

(upper indexed)

- The contra-variant basis vectors transform as a column

$$\vec{e}^{\,i} = (\mathcal{R})^i{}_j \, \vec{e}^{\,j}$$

While the covariant (lower) components transform as a row with the inverse transformation

$$\underline{v}_i = v_i \, (\mathcal{R}^{-1})^i{}_j$$

So that

$\vec{v} = v_i \vec{e}^{\,i}$ is rotationally invariant

<u>Further Examples</u>

Let: $\vec{v} = v^i e_i$

Then

(1) $\vec{e}^{\,j} \cdot \vec{v} = v^i \, \vec{e}^{\,j} \cdot e_i = v^i \, \delta^j{}_i = v^j$

(2) $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

$\vec{e}^{\,i} \cdot \vec{e}^{\,j} = \delta^{ij}$

$\vec{e}^{\,i} \cdot \vec{e}_j = \delta^i{}_j$

(3) $\vec{e}_j \cdot \vec{V} = v^i \, \vec{e}_j \cdot \vec{e}_i = V^i \delta_{ji}$

$$= V_j$$

- <u>Consistency</u>: (where $\mathcal{R}^{-1} = \mathcal{R}^T$)

We said that for rotations that upper and lower are the same. But upper components transform with $\mathcal{R}$ as a column

(☆) $$\underline{v}^i = (\mathcal{R})^i{}_j \, v^j$$

while lower indices transform as row but with the inverse transformation (this is known as the inverse-transpose representation)

(☆☆) $$\underline{v}_i = v_i \, (\mathcal{R}^{-1})^i{}_j$$

Since v_x is the same as v^x they should have the same transformation, indeed since $\mathcal{R}^{-1} = \mathcal{R}^T$

$$\underline{v}_i = v_i \, (\mathcal{R}^T)^i{}_j$$

(☆☆☆) $$\underline{v}_i = (\mathcal{R})_j{}^i \, v_i$$

recall the transpose interchanges columns and rows. We undo the transpose here.

this is the same transformation rule as Eq ☆ above.