1 Introduction to the Maxwell Equations

1.1 The maxwell equations and units

- We use Heavyside Lorentz system of units. This is discussed in a separate note.
- The Maxwell equations are

$$\nabla \cdot \boldsymbol{E} = \rho \tag{1.1}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{1.2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.3}$$

$$-\nabla \times \boldsymbol{E} = \frac{1}{c}\partial_t \boldsymbol{B} \tag{1.4}$$

In integral form we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = Q_{\rm enc} \tag{1.5}$$

$$\oint_\ell \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{I}{c} + \frac{1}{c}\partial_t \Phi_E \tag{1.6}$$

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0 \tag{1.7}$$

$$-\oint_\ell \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c}\partial_t \Phi_B \tag{1.8}$$

Here $\Phi_E = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$ is the electric flux, $\Phi_B = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ is the magnetic flux, and $I = \int_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S}$ is the current crossing a surface, S. $\mathrm{d}\boldsymbol{S}$ is the surface element with a specified area and normal $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d(area)}$. $\mathrm{d}\boldsymbol{\ell}$ denotes a closed line integral element.

- We specify the currents and solve for the fields. In media we specify a constituent relation relating the current to the electric and magnetic fields.
- The Maxwell force law

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \tag{1.9}$$

- Hemholtz Theorems state:

(a) Given a curl free vecor field, $\boldsymbol{C}(\boldsymbol{x})$, there exsists a scalar function, $S(\boldsymbol{x})$, such that $\boldsymbol{C} = -\nabla S$:

$$\text{if} \quad \nabla \times \boldsymbol{C}(\boldsymbol{x}) = 0 \quad \text{then} \quad \boldsymbol{C} = -\nabla S(\boldsymbol{x}) \tag{1.10}$$

(b) Given a divergence free vector field, $\boldsymbol{D}(\boldsymbol{x})$, there exsists a vector field $\boldsymbol{V}$ such that $\boldsymbol{D} = \nabla \times \boldsymbol{V}$:

$$\text{if} \quad \nabla \cdot \boldsymbol{D}(\boldsymbol{x}) = 0 \quad \text{then} \quad \boldsymbol{D} = \nabla \times \boldsymbol{V}(\boldsymbol{x}) \tag{1.11}$$

The converses are easily proved, $\nabla \times \nabla S(\boldsymbol{x}) = 0$, and $\nabla \cdot \nabla \times \boldsymbol{V}(\boldsymbol{x}) = 0$ There are two *very important* consequences for the Maxwell equations.

(a) From the source free Maxwell equations (eqs. three and four) one finds that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.12}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\phi \tag{1.13}$$

(b) Current conservation follows by manipulating the sourced maxwell equations (eqs. one and two)

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0 \tag{1.14}$$

- For a system of characteristic length L (say one meter) and characteristic time scale T (say one second), we can expand the fields in $1/c$ since $(L/T)/c \ll 1$:

$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \dots \tag{1.15}$$

$$\boldsymbol{B} = \boldsymbol{B}^{(0)} + \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \dots \tag{1.16}$$

where each term is smaller than the next by $(L/T)/c$. At zeroth order we have

$$\nabla \cdot \boldsymbol{E}^{(0)} = \rho \tag{1.17}$$

$$\nabla \times \boldsymbol{E}^{(0)} = 0 \tag{1.18}$$

$$\nabla \cdot \boldsymbol{B}^{(0)} = 0 \tag{1.19}$$

$$\nabla \times \boldsymbol{B}^{(0)} = 0 \tag{1.20}$$

These are the equations of electro statics. Note that $\boldsymbol{B}^{(0)} = 0$ to this order (for a field which is zero at infinity)

- At first order we have

$$\nabla \cdot \boldsymbol{E}^{(1)} = 0 \tag{1.21}$$

$$\nabla \times \boldsymbol{E}^{(1)} = 0 \qquad (\text{ since } \partial_t \boldsymbol{B}^{(0)} = 0 \text{ }) \tag{1.22}$$

$$\nabla \cdot \boldsymbol{B}^{(1)} = 0 \tag{1.23}$$

$$\nabla \times \boldsymbol{B}^{(1)} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E}^{(0)} \tag{1.24}$$

This is the equation of magneto statics, with the contribution of the Maxwell term, $1/c\,\partial_t \boldsymbol{E}^{(0)}$, computed with electrostatics. Note that $\boldsymbol{E}^{(1)} = 0$